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On the Motions of Ships in Confused Seas[1]

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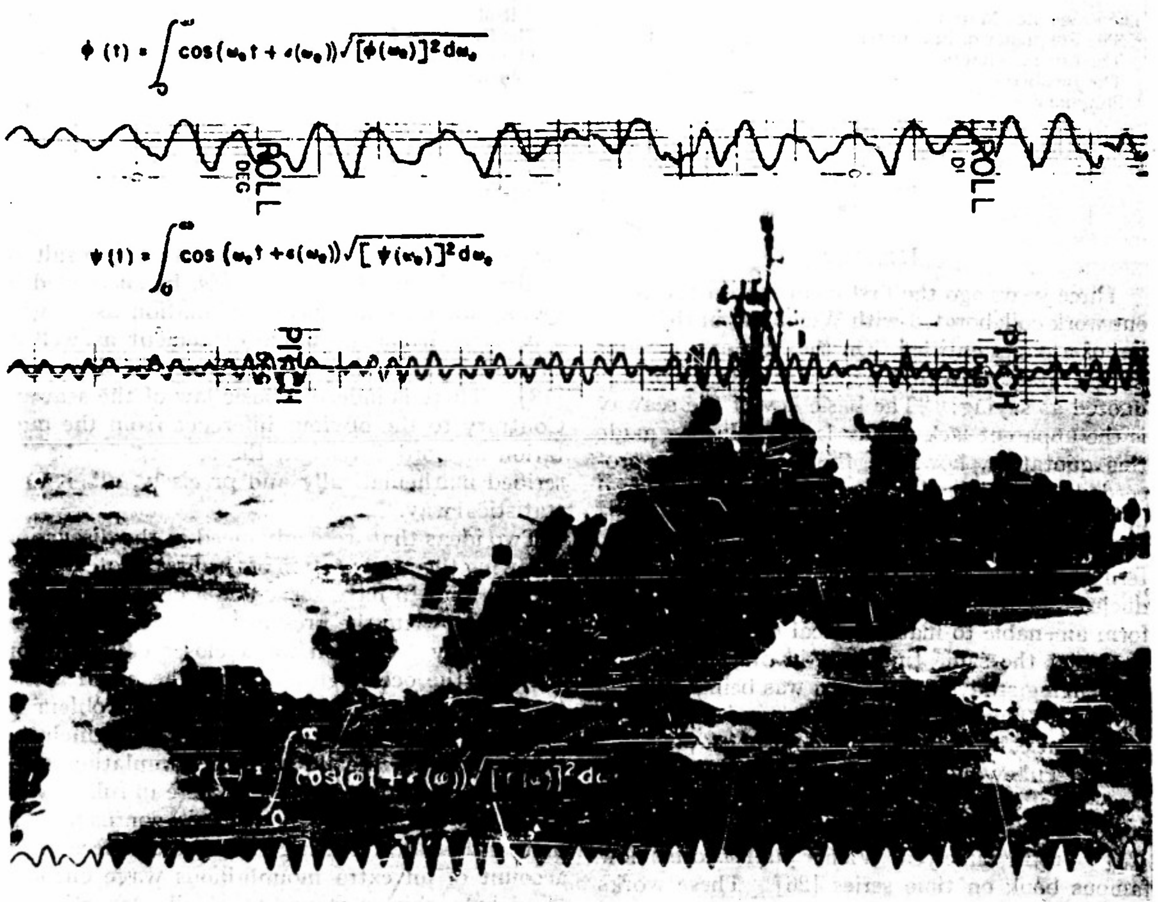

"Mathematics Can Never Tell You What Is; Only What Would Be If." —Poincaré

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Topical Outline

INTRODUCTION

History

Three years ago the first co-author of the present work collaborated with Weinblum in the writing of a paper entitled "On the Motions of Ships at Sea" [24].[4] In that paper Lord Rayleigh was quoted as saying: "The basic law of the seaway is the apparent lack of any law." Having made this quotation, however, the authors then proceeded to consider the seaway as being composed of "a regular train of waves defined by simple equations." This artificial substitution of pattern for chaos was dictated by the necessity of reducing the utterly confused reality to a simple form amenable to mathematical treatment.

Yet at the same time and in other fields the challenging study of confusion was being actively pursued. Thus in 1945 Rice was writing on the mathematical analysis of random noise [18] and in 1949 Tukey and Hamming were writing on the properties of stationary time series and their power spectra in connection with colored noise [23]. In the same year Wiener published his now famous book on time series [26]. These works were written as contributions to the theory of communication. Nevertheless the fundamental mathematical discipline expounded therein can readily be extended to other fields of scientific endeavor. Thus in 1952 the second co-author, inspired by a contribution of Tukey, was able to apply the foregoing theories to the study of actual ocean waves [22] [14]. As the result of analyses of actual wave records, he succeeded in giving not only a logical explanation as to why waves are irregular, but a statement as well of the laws underlying the behavior of a seaway [13]. There is indeed a basic law of the seaway. Contrary to the obvious inference from the quotation of Lord Rayleigh, the seaway can be described mathematically and precisely, albeit in a statistical way.

Two ideas that were advanced in the discussion following the presentation of the first quoted paper may be said to have served as a challenge to the authors to write the present work. The first was the plea by Davidson for a closer collaboration between the oceanographer and the naval architect for the purpose of treating the problem of the actual irregular surface of the sea on which the ship sails. The second was the stimulating comment by Wheelock reproduced here in full: "It is axiomatic that a confused sea is confusing. A matter that always intrigues me is the occasional account of an extra mountainous wave encountered by a ship, a wave out of all proportion to the others in the area. Since the sea surface is apparently the sum of a multiplicity of simple wave systems, it would be enlightening to learn what would be the result of a concurrence of all crests of an assumed set of wave systems, the time probability of such an event and the probability of a ship under way in the area encountering one. After all we must provide for the worst."

[4] Numbers in brackets refer to references listed at the end of this paper.

Purpose

The purpose of this paper is to integrate the results of Weinblum and St. Denis, of Rice, of Tukey and Hamming and of Pierson in the endeavor to satisfy the comments made in 1950. To this end a rigorous statistical definition will be given of sea conditions: one which will bring out the basic difference between a "sea" and a "swell." The thesis will be then to derive the response of a ship to confused seas, whether these be composed of storm waves or of swells. The authors' intent is to present findings useful to the designer and researcher engaged in the study of ship motions.

Because of parallel studies presently underway on wave generation and decay, it is now possible to forecast wave conditions several days in advance of their occurrence. When an integration is made of the studies on the forecasting of sea conditions and those on ship response, the results may well prove to be of importance to the navigator as well.

The sea loses little of its mystery by being defined. Nevertheless, it is essentially on the basis of the definition to be given herein that it will be possible to arrive at some prediction of the motions of a vessel tossed about by the fury of the sea. To the extent that the naval architect learns to foretell the behavior of a ship in her natural element, he acquires in her a laboratory with which to gain further insight into the natural phenomena of his concern.

It is hoped that this paper will be the first in a long series of works of collaboration between naval architect and oceanographer. They have, indeed, much to share with each other and much to tell jointly. For the sister professions they serve have one great element in common: the inconstant and boundless sea.

Coordinate Systems

Before entering into an exposition of the theory, the space coordinate systems to be used will be defined. There are four such systems: all are right-handed and so directed that the Z axis is vertically upward. In each system angular measurements are made counterclockwise with reference to the positive X axis. In the major part of the derivations of this paper, coordinate systems (c) and (d) listed below will be used. The mutual relations of all the systems herein employed are given in Fig. 1.1 and Table 1.1 gives a summary of the transformations of each system into the others.

(a) *Fixed Coordinate System Oriented with Respect to the Earth.* The coordinates may be identified by the subscript a, for absolute. The origin of this system is at a fixed point on the surface of the sea. The positive X_a and Y_a axes are due east and due north, respectively. The angles χ_a are measured counterclockwise from the positive X_a axis. This system is used in describing the seaway *per se* without reference to a vessel's position and course.

Table 1.1.—Transformations of Systems

	Absolute	Wind	Relative	Vessel[a]
Abs.	....	$X_w = X_a \cos\theta_w + Y_a \sin\theta_w$ $Y_w = -X_a \sin\theta_w + Y_a \cos\theta_w$ $\chi_w = \chi_a - \theta_w$	$X = X_a \cos\theta + Y_a \sin\theta$ $Y = -X_a \sin\theta + Y_a \cos\theta$ $\chi = \chi_a - \theta$	$X_e = X_a \cos\theta + Y_a \sin\theta - vt$ $Y_e = -X_a \sin\theta + Y_a \cos\theta$ $\chi_e = \chi_a - \theta$
Wind	$X_a = X_w \cos\theta_w - Y_w \sin\theta_w$ $Y_a = X_w \sin\theta_w + Y_w \cos\theta_w$ $\chi_a = \chi_w + \theta_w$	....	$X = X_w \cos(\theta - \theta_w) - Y_w \sin(\theta - \theta_w)$ $Y = X_w \sin(\theta - \theta_w) + Y_w \cos(\theta - \theta_w)$ $\chi = \chi_w + (\theta - \theta_w)$	$X_e = X_w \cos(\theta - \theta_w) + Y_w \sin(\theta - \theta_w) - vt$ $Y_e = -X_w \sin(\theta - \theta_w) + Y_w \cos(\theta - \theta_w)$ $\chi_e = \chi_w + (\theta - \theta_w)$
Rel	$X_a = X \cos\theta - Y \sin\theta$ $Y_a = X \sin\theta + Y \cos\theta$ $\chi_a = \chi + \theta$	$X_w = X \cos(\theta_w - \theta) + Y \sin(\theta_w - \theta)$ $Y_w = -X \sin(\theta_w - \theta) + Y \cos(\theta_w - \theta)$ $\chi_w = \chi - (\theta_w - \theta)$	....	$X_e = X - vt$ $Y_e = Y$ $\chi_e = \chi$
Vessel	$X_a = X_e \cos\theta_e - Y_e \sin\theta_e + vt \cos\theta_e$ $Y_a = X_e \sin\theta_e + Y_e \cos\theta_e + vt \sin\theta_e$ $\chi_a = \chi_e + \theta_e$	$X_w = X_e \cos(\theta_w - \theta_e) + Y_e \sin(\theta_w - \theta_e) + vt \cos(\theta_w - \theta_e)$ $Y_w = -X_e \sin(\theta_w - \theta_e) + Y_e \cos(\theta_w - \theta_e) + vt \sin(\theta_w - \theta_e)$ $\chi_w = \chi_e - (\theta_w - \theta_e)$	$X = X_e + vt$ $Y = Y_e$ $\chi = \chi_e$	

[a] Note $\theta_e \equiv \theta$

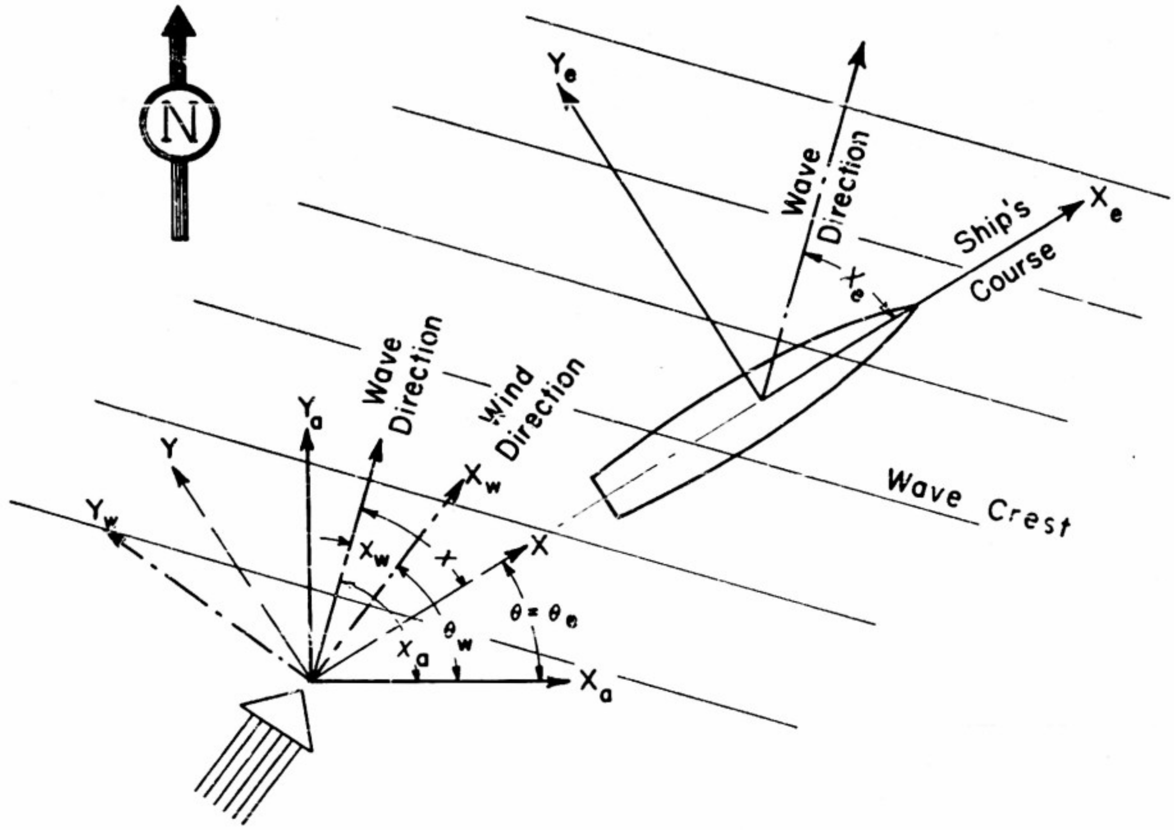


FIG. 1.1.—COORDINATE SYSTEMS

(b) *Fixed Coordinate System Oriented with Respect to the Wind Direction.* The coordinates may be identified by the subscript w, for wind. The origin of this system is the same as the preceding. The positive X_w axis is in the direction toward which the wind is blowing. This system is used in obtaining the frequency spectrum of the seaway.

(c) *Fixed Coordinate System Oriented with Respect to the Moving Ship.* The coordinates may be identified by the absence of any subscript since this will be the most frequently used system. The origin of this system is the same as the two preceding. The positive X axis is in the direction of the ship's heading. This system is used to obtain the response amplitude operators.

(d) *Moving Coordinate System Oriented with Respect to the Vessel.* The coordinates may be identified by the subscript e, for effective or encounter. The origin of this system is at the vessel's center of mass. The positive X_e axis is in the direction of the ship's heading. This system is used to obtain the ship's response.

PROCEDURE

In this paper the pattern of presentation of the theory will follow an order of cause and effect. For the subject matter this is written

Seaway $\times$ Response amplitude operator $\times$ Frequency mapping = Ship response

The seaway is first obtained with reference to the direction of the wind, or winds, generating it. When the ultimate objective is simply the definition of the sea state, then the seaway is most conveniently defined with reference to an absolute system of coordinates. This is apparent if it is reflected that a seaway is often made up of wave trains originating in more than one area. When the ultimate objective is the response of a ship, then the seaway needs be defined with reference to a relative coordinate system. Up to this point the transformations required to change the coordinate systems are simple: they involve merely rotations about a fixed point. However, the motions of a ship must be ultimately referred to the

moving coordinate system of the vessel. As will be seen later, this step introduces some complexities and results in a complete change in the pattern of wave frequencies. It appears convenient to change the reference system from one which is fixed in space (system c) to one which is moving with the vessel (system d) at that step in the procedure which considers the frequency mapping. The following then, is the order of steps in developing the theory of this paper:

(a) Definition of the seaway—with reference to the fixed coordinate systems (absolute, wind and relative).

(b) Calculation of the response amplitude operators—with reference to the relative system.

(c) Mapping of the frequencies—from the relative to the vessel's system.

(d) Determination of the ship's response—with reference to the vessel's coordinate system.

THE SEAWAY

"In a broad sense the laws of nature are Gaussian."

Preliminary Remarks

When a storm is raging at sea the waves generated by the wind present a picture of endless confusion from which any sense of regularity is totally absent. Only as the waves travel far from the storm area where they were generated do they acquire some apparent periodicity or rhythmic pattern. As the distance from the generating area increases this aspect of regularity becomes more pronounced. Complete regularity in the rise and fall of the sea surface is never attained in nature. Where, as in the case of a running swell, it is approached, the waves have become so reduced in height that this case is not of dominant interest in the application to ship motions. Yet almost all studies of ship motions have been based on a sea of perfect regularity, in spite of the obviously restricted applicability of the ensuing results. No dependable estimates can be derived from such theories as to the motions, accelerations and drop in speed that a vessel may experience in heavy seas. As a consequence, a reliable guide is lacking for assessing the relative merits of comparative hull forms.

Because of the powerfully limited applicability of the theory on ship motions based on regular (long-crested, sinusoidal) seas, the authors have inquired as to how far a theory could be developed for predicting ship motions in completely irregular seas. The object of this paper is to present such development for criticism by the profession.

The results of this paper will show that if it is possible to determine the response of a ship to a simple sinusoidal wave system of any period and direction, then essential properties of the response of a ship to any actual seaway, however confused, can also be found. There is thus made available a means of predicting the amplitude of ship motions with a greater reliability than possible heretofore.

In order to lead up to the representation of the seaway, advanced herein, it is desirable to discuss first other possible mathematical models for representing a complex seaway and the limitations to which each is subjected.

To provide a logical point of departure, some known expressions of the classical wave theory will first be recalled.

Classical Wave Theory

The theory of gravity waves presently in use is normally based on the assumption that the height of the waves is small. This assumption permits a linearization to be made which, in the present application, will prove to be of essential importance in describing the seaway. The derivation of this theory, as given in Lamb, remains one of the most brilliant achievements in classical hydrodynamics [7].

In accordance with this theory, a surface disturbance, as created by a periodic impulsive force acting along a straight line, travels from the source to infinity as a progressive wave system. Each wave of this system is sinusoidal in profile, has an amplitude unvarying from that of all its predecessors and successors and requires exactly the same increment of time to pass by a given point. When referred to a fixed system of coordinates (X, Y either absolute, relative or wind) and the time variable, t, the free surface of the wave is completely described by the expression

$$r(X, Y, t) = r_m \cos\left[\frac{\omega^2}{g}(X\cos\chi + Y\sin\chi) - \omega t + \epsilon\right] \quad (1.1)$$

In this equation $r(X, Y, t)$ is the height of the sea surface above or below the X-Y plane ($Z = 0$), r_m is the wave amplitude, ω is the angular wave frequency such that $\omega = 2\pi/\tau$, where τ

is the wave period, χ is the direction of wave propagation and ϵ is any phase angle. The choice of value for the last symbol is arbitrary: it determines the phase for the wave system at any point at the instant $t = 0$. The values r_m, χ, ω and ϵ are all fixed quantities for a particular sinusoidal wave system. In infinitely deep water, which for any practical purpose is water of a depth greater than half of a wave length, the length of wave, λ, and its velocity of travel, c, are simply functions of the wave period and the various relationships given hereunder obtain:

$$c = \frac{\lambda}{\tau} = \frac{g}{\omega} = \sqrt{\frac{g\lambda}{2\pi}} = \frac{g\tau}{2\pi} \tag{1.2}$$

$$\lambda = \frac{2\pi c^2}{g} = \frac{2\pi g}{\omega^2} = \frac{g\tau^2}{2\pi} \tag{1.3}$$

At times in current literature use is made of a wave number defined as[5]

$$k \triangleq \frac{2\pi}{\lambda} = \frac{\omega^2}{g} = \frac{g}{c^2} = \frac{4\pi^2}{g\tau^2} \tag{1.4}$$

The sea surface defined by equation (1.1) resembles at any instant of time a sheet of corrugated material such as is used for roofing purposes. The wave crests are infinitely long straight lines and their spacing is given by equation (1.3).

Representation of the Seaway at a Fixed Point—The Origin

The problem of defining the seaway as a function of time over an extent of the sea surface may be worded as follows:

If the sea surface $r(X, Y, t)$ is known as a function of time at a given point (X_0, Y_0), what will it be at any other point (X, Y)? In symbolic representation: Given $r(X_0, Y_0, t)$ determine $r(X, Y, t)$.

As a first step the problem is reduced to that of representing the free surface at a given point as a function of time. Without loss of generality, the given point may be chosen as the origin of a fixed system of coordinates. Symbolically then, this first step is that of representing $r(0, 0, t)$. With an obvious simplification of notation, the problem is stated simply to be that of determining $r(t)$, the rise and fall of the surface of the sea at the origin of a fixed system of coordinates.

Over time intervals of the order of days $r(t)$ at any fixed point is not even remotely periodic. The amplitude of the waves may vary from negligibly small values to what may be referred to as mountainous proportions. Fortunately, in the study of ship motions such a radical change in sea state is not of immediate interest and the problem can be reduced to that of analyzing $r(t)$ under the assumption that some property thereof is preserved for a reasonable time interval. The reservation is made that the situation is still undefined outside of some possibly larger time interval. What does constitute a reasonable time interval will depend on the intended application. It may be observed, however, that the sea state changes very slowly and can be assumed to remain essentially constant for periods of the order of 20 to 30 minutes at least.

Thus, in studies of ship motions it appears reasonable to assume a steady state of the sea.

Consider, then, a wave record of some 20 minutes duration. Is it possible to derive some functional description for $r(t)$ which will represent the wave record for the given period?

One may begin by considering the not-too-irregular wave record of Fig. 1.2.

(1) *Periodic Wave System with an Amplitude Component at a Single Spectral Frequency.* In a theory developed by Sverdrup and Munk for defining actual ocean waves, wave records are analyzed on the basis of significant height and period [21]. In this analysis significant height and period are defined as the average of those obtained upon consideration of the one-third highest waves. Thus a regular pattern represented by

$$r(t) \cong r_m \cos\left(\frac{2\pi t}{\tau} + \epsilon\right) = r_m \cos(\omega t + \epsilon) \tag{1.5}$$

is made to replace the irregular trace of the waves. This representation implies that the spectrum of the wave amplitude is concentrated at a single value of the frequency, Fig. 1.3. It also implies that the wave record repeats itself every τ seconds and that the wave amplitude is constant.

If the representation of the wave amplitude given by equation (1.5) is compared to the actual wave record, the two being so adjusted that coincidence is obtained for $t = 0$, the following features become evident:

(a) Although the two records are in apparent phase at $t = 0$, they soon are out of phase.

(b) The heights of the two wave records will rarely coincide. About five-sixths of the time the actual wave heights will be found to be lower than the significant height of the periodic wave system used in the representation.

Since by this representation the actual wave record is replaced by a purely periodic function with one discrete spectral component, there are only two parameters that can be chosen. These two parameters do not describe adequately the actual wave record as a function of time.

[5] The sign $\triangleq$ means equals by definition.

FIG. 1.2.—ACTUAL WAVE RECORD

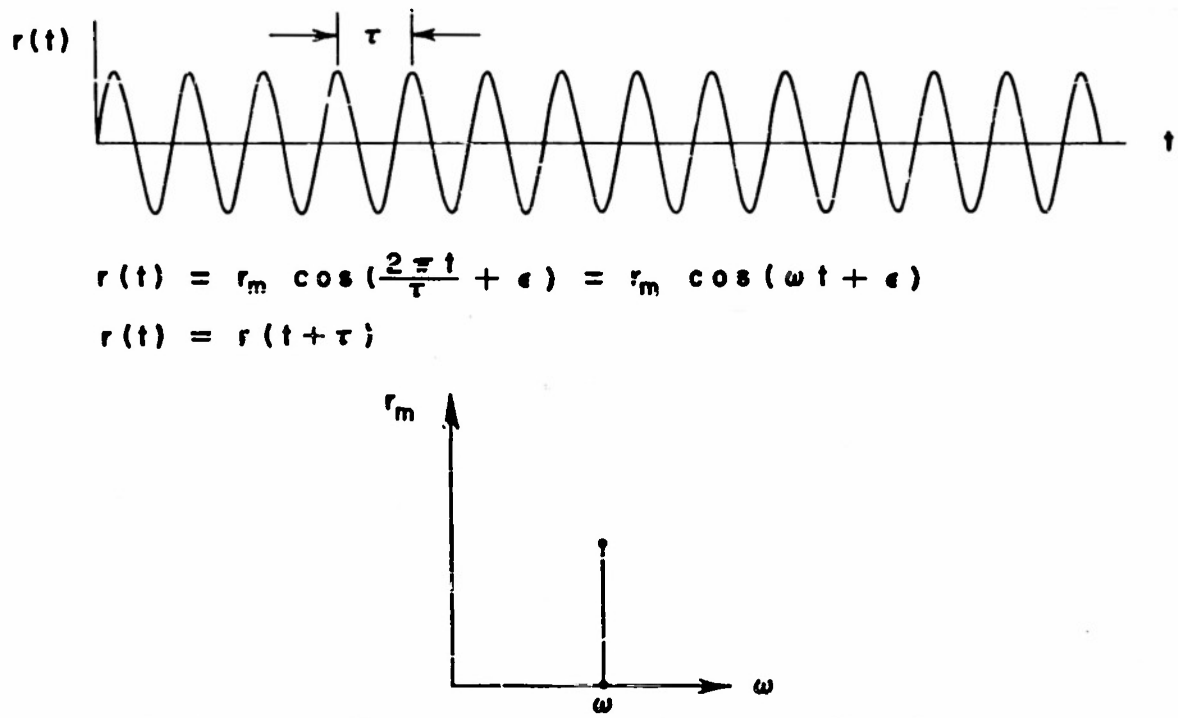


FIG. 1.3.—PERIODIC WAVE SYSTEM WITH AN AMPLITUDE COMPONENT AT A SINGLE SPECTRAL FREQUENCY

(2) *Periodic Wave System with Amplitude Components at Many Discrete Spectral Frequencies.* A second way to analyze the actual wave record of Fig. 1.2 is to consider the wave pattern extending from say $t = 0$ to $t = t_1$ and assume that this pattern repeats itself exactly every t_1 seconds. If then the particular pattern is analyzed by a Fourier series the following representation is obtained:

$$r(t) \cong \sum_{n=1}^{\infty} r_n \cos\left(\frac{2\pi n t}{t_1} + \epsilon_n\right)$$
$$= \sum_{n=1}^{\infty} r_n \cos(\omega_n t + \epsilon_n) \quad (1.6)$$

The discrete spectral wave periods which determine the wave components of the series are found by dividing the period of repetition t_1 by the natural series of integers. A frequency spectrum for this representation is indicated in Fig. 1.4.

If the actual wave record were to be expanded in a Fourier series the following features would be noted:

(a) The lower harmonics would all be negligible. The number of such negligible harmonics would depend on the duration of the record chosen t_1. Since the amplitude of a harmonic component does not ordinarily become appreciable until its period is less than some 25 seconds, the num-

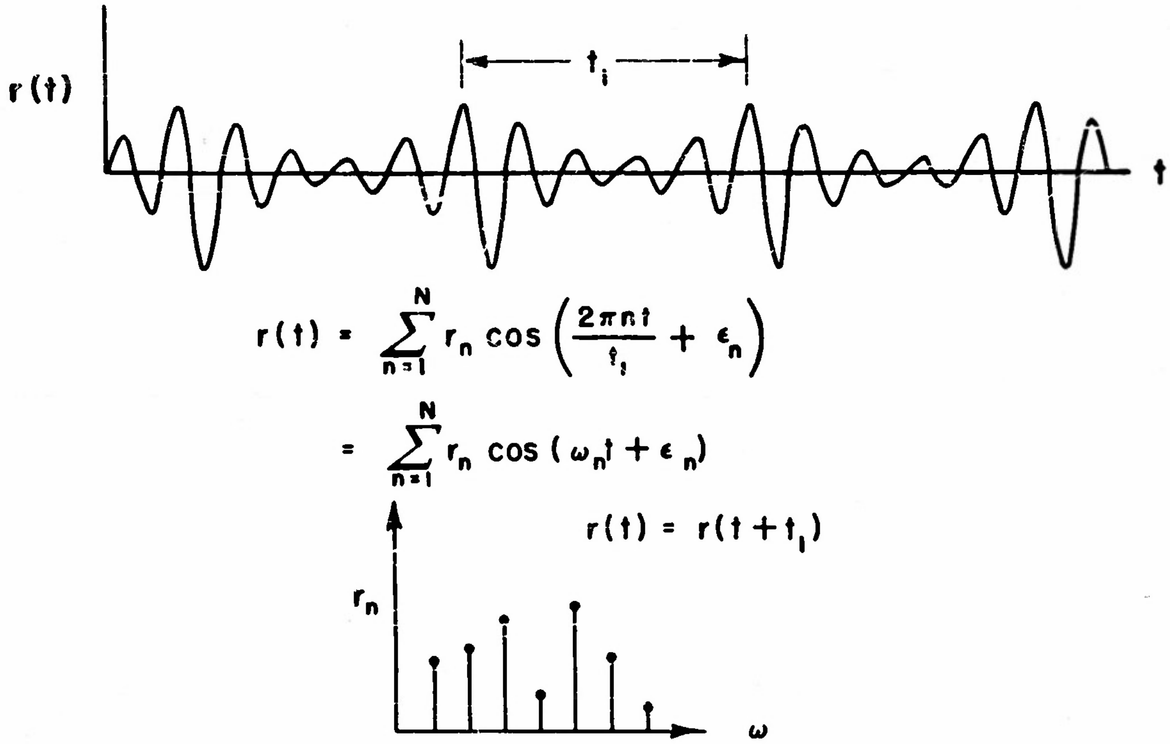


FIG. 1.4 PERIODIC WAVE SYSTEM WITH AMPLITUDE COMPONENTS AT MANY DISCRETE SPECTRAL FREQUENCIES

ber of negligible components would be given approximately by $n \cong t_1/25$.

(b) The harmonics of maximum amplitude would correspond to those having a period somewhat greater than that of the significant waves in the group.

(c) Those higher harmonics whose period is less than some three seconds would also be negligible.

(d) The values of the r_n's would be highly erratic and rapidly varying for successive values of n, see Rice [18].

Thus, if t_1 were chosen to be 100 seconds, the first four harmonics and all harmonics above perhaps the thirtieth would be neglected. If t_1 were chosen to be 20 minutes, then the harmonics would not become appreciable until approximately the fiftieth term and they would not cease to be significant until perhaps the four hundredth term.

The work of obtaining a Fourier expansion would be extremely tedious and would bring but meager rewards, for the results would not be too amenable to theoretical work. But such an expansion would serve to bring out one fact, namely, that intervals of time of low wave height in the record are caused essentially by the phase cancellation of a large number of harmonic components of low amplitude whereas intervals of time of large wave height result from the phase reinforcement of the same harmonic components.

The Fourier expansion discussed would be a true representation of the actual wave record for $0 < t < t_1$, but would fail to represent the same record beyond this period, i.e., for $t < 0$ and for $t > t_1$, because the mathematical model would repeat itself periodically over intervals of length t_1.[6]

(3) *Aperiodic Wave System Having a Continuous Amplitude Spectrum—Representation by the Fourier Integral Theorem.* A third possible way to analyze an actual wave record is by application of the Fourier Integral Theorem. To this end the wave record extending from $t = 0$ to $t = t_n$ is divided into a number of regions $(0 < t < t_1)$, $(t_1 < t < t_2) \ldots (t_{n-1} < t < t_n)$ and the wave record is separately defined in each region. Each separate definition is valid only within the boundaries of the region wherein it applies and identically zero outside thereof.

$$r(t) = r_0(t) + r_1(t - t_1) + \ldots r_{n-1}(t - t_{n-1}) \quad (1.7)$$

[6] Of course this statement does not hold for all possible mathematical models having a discrete spectrum. A series having spectral components at irrational values of the frequency does not repeat itself.—See H. Bohr, "Almost Periodic Functions," Chelsea Publishing Co., New York.

In the region $(t_p < t < t_{p+1})$, where $0 < p < n - 1$, the Fourier Integral expansion is given by

$$r_p(t) = \int_0^\infty a_p(\omega) \cos \omega t \, d\omega + \int_0^\infty b_p(\omega) \sin \omega t \, d\omega \qquad (1.8)$$

where

$$a_p(\omega) \triangleq \frac{1}{\pi} \int_{-\infty}^{\infty} r_p(t) \cos \omega t \, dt \qquad (1.9a)$$

$$b_p(\omega) \triangleq \frac{1}{\pi} \int_{-\infty}^{\infty} r_p(t) \sin \omega t \, dt \qquad (1.9b)$$

The amplitude of each spectral component of the wave is a continuous function of the frequency and is given by $\sqrt{a_p^2 + b_p^2}$.

For each division of the wave record the spectral representation is continuous as indicated in Fig. 1.5 and from this representation the relative importance of any part of the spectrum is manifest.

In this manner $r(t)$ can be made to represent the actual wave record over any length of time chosen for analysis and this representation would be exact for the interval analyzed. It would fail, however, for time outside the interval in which the analysis was performed, since the record would either have to be defined as identically equal to zero or would remain unknown.

One disadvantage of this method is that there is no known precise procedure by which, starting with a wave record, it is possible to determine the appropriate $a_p(\omega)$ and $b_p(\omega)$. Nevertheless, such a procedure is theoretically possible. The Fourier Integral method would be a convenient one for studying problems in transient response. It does not, however, lend itself to problems of the seaway because of the extremely long duration of the records to be analyzed.

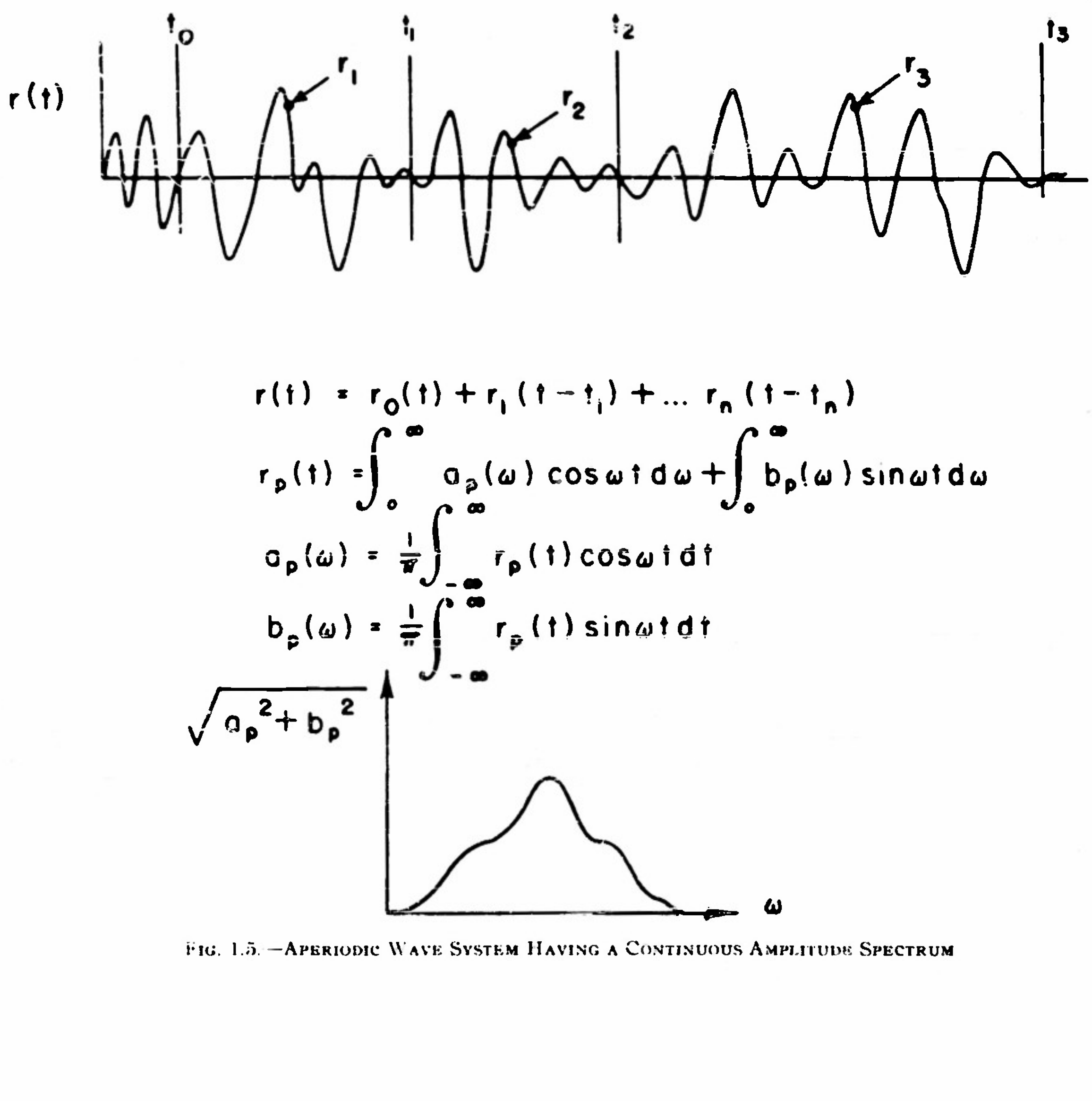


FIG. 1.5.—APERIODIC WAVE SYSTEM HAVING A CONTINUOUS AMPLITUDE SPECTRUM

(4) *Aperiodic Wave System Having a Continuous Energy Spectrum—Representation by the Energy Integral for the Gaussian Case.* All the mathematical models described above prove inadequate to represent in a practical manner a wave record whose basic feature is its irregularity. It follows that what is needed is a description of the wave state at a point which will be realistic and readily handled in the problems to which it is applied, no matter how great the complexity of the sea. Such a fourth representation is provided in the form of the energy integral for the Gaussian case.

For additional details on this integral which is employed in theoretical statistics, the interested reader is referred to the previously quoted works by Tukey, by Tukey and Hamming, by Rice and by Wiener. Additional references by Lévy and by Cramér may also be listed [8] [5]. These sources are not exhaustive. They do contain, however, an exposition of methods which can be used directly in wave analysis and a wealth of information on the statistical properties of the representation.

According to this representation the wave state at a fixed point is given by

$$r(t) = \int_0^\infty \cos\,[\omega t + \epsilon(\omega)]\cdot\sqrt{[r(\omega)]^2\,d\omega} \quad (1.10)$$

The integral of this equation is not an integral in the ordinary sense: It cannot be integrated formally. It represents simply a mathematical abstraction which, however, can be approximated to any degree of accuracy by a partial sum. But before discussing this approximation it is necessary first to introduce the concepts of "energy spectrum," "cumulative energy density" and "random phase."

(*a*) *Energy Spectrum.* This is represented by the symbol $[r(\omega)]^2$ in equation (1.10). It is shown as a squared function of ω to indicate that it is everywhere positive. It is used only as a squared quantity and has the dimensions of a squared length multiplied by time. It is not to be confused with $r(t)$ which represents the wave record as a function of time. It measures the average squared value of the wave amplitude associated with a particular frequency ω. It is obvious, therefore, that the energy spectrum differs with the condition of the sea. The term energy spectrum is used because of the intimate connection of $[r(\omega)]^2$ with the average potential energy of the waves on the surface of the sea.

(b) *Cumulative Energy Density.* The integral over ω of the energy spectrum yields a bounded value having the dimensions of a length squared

$$\int_0^\omega [r(\omega)]^2\,d\omega \triangleq R(\omega) \quad (1.11)$$

This value, known as the "cumulative energy density," measures that part of the averaged square value of $r(t)$ which is contributed by those spectral frequencies less than or equal to ω. The cumulative energy density for the full spectral range

$$\int_0^\infty [r(\omega)]^2\,d\omega \triangleq R(\infty) \triangleq R \quad (1.12)$$

may be used as a convenient parameter to describe the sea state.

(c) *Random Phase.* In equation (1.10) this is represented by the symbol $\epsilon(\omega)$. It is a so-called point set function whose values are random and equally probable for any value of between 0 and 2π. In computing the integral of equation (1.10) $\epsilon(\omega)$ is to be assigned values at certain given frequencies in accordance with the equation

$$p\,[0 < \epsilon(\omega) < 2\pi\alpha] = \alpha \quad (1.13)$$

which is to be read as follows: The probability that $\epsilon(\omega)$ will lie between the values of zero and $2\pi\alpha$ is equal to α. Here α ranges from zero to one. Hence all phases are equally probable. Also all phases are independent.

The random phase is nowhere continuous and cannot be graphed. Nevertheless, it does not lack definition.

The integration of equation (1.10) is not a simple process because of the infinite number of discontinuities introduced by the randomly chosen phase angles.

The integral of expression (1.10) is obtained as the limit of a sum in the following manner:

(a) Divide the range of integration (ω axis) by a series of arbitrarily spaced points: 0, ω_1, . . . ω_n . . . ω_{2q}, forming a one-dimensional net.

(b) Obtain $[r(\omega)]^2$ for all alternate (odd-numbered) points $2n + 1$.

(c) Obtain interval $(\omega_{2n+2} - \omega_{2n})$.

(d) Calculate $\sqrt{[r(\omega_{2n+1})]^2\cdot(\omega_{2n+2} - \omega_{2n})}$.

(e) Select $\epsilon(\omega)$ at random for each ω_{2n+1}, obtaining $\epsilon(\omega_{2n+1})$ and calculate $\cos\,[\omega_{2n+1}t + \epsilon(\omega_{2n+1})]$. The random selection of $\epsilon(\omega)$ is to be made in accordance with equation (1.13).

(f) Form the partial sum

$$\sum_{n=0}^{q} \cos\,[\omega_{2n+1}\,t + \epsilon(\omega_{2n+1})]\cdot \sqrt{[r(\omega_{2n+1})]^2\,(\omega_{2n+2} - \omega_{2n})}$$

The limit of this sum as the interval $(\omega_{2n+2} - \omega_{2n})$ of the net approaches zero and as the range ω_{2q} approaches infinity is the wave record $r(t)$, i.e.,

$$r(t) = \lim_{\substack{(\omega_{2n+2} - \omega_{2n}) \to 0 \\ \omega_{2q} \to \infty}} \sum_{n=0}^{q} \cos\left[\omega_{2n+1}t + \epsilon(\omega_{2n+1})\right] \cdot \sqrt{[r(\omega_{2n+1})]^2 (\omega_{2n+2} - \omega_{2n})}$$

$$\doteq \int_0^\infty \cos\left[\omega t + \epsilon(\omega)\right] \sqrt{[r(\omega)]^2\, d\omega} \quad (1.14)$$

In actual application the interval can be made constant and small but finite, and the range can be made large but also finite. A net can always be made of so small a mesh that the partial sum becomes indistinguishable, in a practical sense, from the actual wave record it approximates. The wave state at a point is thus written

$$r(t) \cong \sum_{n=0}^{t} \cos\left[\omega_{2n+1} t + \epsilon(\omega_{2n+1})\right] \cdot \sqrt{[r(\omega_{2n+1})]^2 (\omega_{2n+2} - \omega_{2n})} \quad (1.15)$$

This partial sum obviously depends on the sequence in which the values for the random phase are chosen by the probability law given above. With a fixed energy spectrum *a whole class of functions $r(t)$ still results.* Several questions naturally arise, to wit: What properties do all these functions have in common? Are the common properties sufficient to allow for a quantitative analysis? And if a part of one function of this class be given, how can the energy spectrum $[r(\omega)]^2$ be found? The first two questions will be answered immediately below. The answer to the third will be delayed until the problem of representing the waves over an extent of sea has been considered.

All these questions can be answered *only in a statistical sense.* Nevertheless, this representation provides the only means to describe realistically the actual surface of the sea.

Because the phase $\epsilon(\omega)$ is chosen at random, the first property of the given representation of the wave state as a function of time at a fixed point is that points picked from a record thereof are distributed according to the Gaussian (or normal) probability law as shown by Rice and Lévy (18), (8).

$$p[k_1 < r(t_1) < k_2] = \frac{1}{\sqrt{\pi R}} \int_{k_1}^{k_2} e^{-k^2/R}\, dk \quad (1.16)$$

This expression is to be read as follows: The probability that at a time t_1, chosen at random, the ordinate of the sea surface at a fixed point will be between k_1 and k_2 is given by the integral of the normal probability distribution from k_1 to k_2.[7]

[7] Equation (1.16) is a very elementary statement of the Gaussian property of the record. In a more refined analysis it could be shown that the values, $r(t_1)$, $r(t_2)$, $r(t_3)$ . . . $r(t_p)$ from such a record are distributed according to a multivariate normal law with p variables and that the correlation between $r(t_j)$ and $r(t_k)$ depends on the difference, $t_j - t_k$. The correlation is given by the value of the (normalized) auto-correlation function of $r(t)$ as given by an appropriate modification of equation (5.2) with h equal to the absolute value, $|t_j - t_k|$.

The second property of such a representation is that the wave record ***never*** repeats itself.

This second property is one of the most important attributes of ocean waves. Yet, with the exception of the representation based on the Fourier Integral Theorem, none of the other mathematical models exhibit this property. The Fourier Integral representation does so, but in an abstract sense. To obtain the functions involved, however, is not a practical problem. Nevertheless, on the basis that points taken from a wave record have a Gaussian distribution, a statistical approach becomes feasible, and clumsiness is replaced by elegance.

The energy integral representation of the wave record lasts forever. Its statistical properties never change. Some part of some one of the limit functions represented by equation (1.14) will be an exact duplicate of a given record. This mathematical model, moreover, still represents in a statistical sense the wave record for times both before and after the duration of the record.

Equation (1.14) is capable of representing the conditions of the seaway for many hours if the winds remain constant in velocity over a large area for a long time. The statistical properties of the record are completely described by the energy spectrum and, by considering the nature of *all* possible such records, the scope of the investigation of a particular record is broadened immensely.

Whether points taken from a record of sea waves actually do follow a Gaussian distribution can be immediately ascertained from wave records. The method of verification (Chi-Square test) will not be discussed here. It may suffice to state that extensive analysis of wave records taken at sea and along the coast indicates that the Gaussian distribution is closely approached except for very high waves for which the non-linear effects become important. Yet, even in this case, the approximation obtained on the assumption that the points of a record follow a Gaussian distribution at present leads further than any other known approach.

The Gaussian property of wave records was first shown by Rudnick [19]. It has been verified by many other observations since then as indicated by Pierson and Marks [14].

The departure of actual wave records from the Gaussian case can be explained on the basis of the actual non-linearity of the sea surface. This non-

linearity is discussed by Lamb [7] p. 417 et seq. Because of it, crests are higher and troughs are shallower than in the hypothetical surface described by the Gaussian case.

The observation naturally arises as to whether the assumption of Gaussian distribution will constitute a serious hindrance to the analysis of actual wave records and ship motions which will be non-Gaussian to a lesser or greater extent. The belief is expressed that it will not. Because the wave systems are non-linear in their high-frequency components, to which a vessel is less responsive, this assumption should be even less serious in the application to ship motions. In any case, there is at best only a remote possibility of obtaining at present a practical non-linear (and consequently non Gaussian) solution to the representation of a seaway. It may be remarked incidentally that even a non Gaussian distribution for the linear case involves great theoretical difficulties for little, if any, added realism. Extensive application of this method strengthens the belief that the most general and realistic way to represent the surface of the sea at a fixed point is given by the Gaussian case of the energy integral.

Upon comparing the simplest representation of the seaway, equation (1.5), and the most realistic, equation (1.10), the following differences are noted:

(a) The single periodic wave system of finite amplitude r_m is replaced by the integral of an infinite number (in actual application, by the sum of a large number) of wave systems of varying frequency, ω, and of infinitesimal (in actual application very small) amplitude given by $\sqrt{[r(\omega)]^2 d\omega}$ or in actual application, by $\sqrt{[r(\omega_{2n+1})]^2(\omega_{2n+2} - \omega_{2n})}$.

(b) The constant phase lag, ϵ, of the single periodic wave system is replaced by an infinity (in actual application by a large number) of phase lags, $\epsilon(\omega)$, each chosen at random and applied to one of the infinitesimally (in actual application, diminutively) high wave systems.

Given a wave record as a function of time, an estimate of the energy spectrum, $[r(\omega)]^2$, can be found by the numerical methods given by Tukey and by Tukey and Hamming. An application of these methods to the analysis of ocean waves has been made in the previously quoted paper by Pierson and Marks. The procedure to be used will be outlined in the last section of this paper.

Representation of the Seaway Over an Area—Fixed System of Coordinates

Up to this point one has been concerned with obtaining a realistic representation of the rise and fall of the sea at the origin of a fixed coordinate system. No sense of wave direction was involved. The last given representation of the sea state is valid even if the wave components come simultaneously from all four quadrants of the compass.

The original problem will now be discussed, namely, that of extending the description of the sea state to all points of the X-Y plane. Without loss of generality, the problem is stated symbolically as follows: Given $r(0, 0, t)$ find a representation for $r(X, Y, t)$.

(1) *Long-Crested Waves.* So long as consideration is restricted to infinitely long-crested waves, this extension is simply accomplished. Since the direction of travel, χ, is the same for all waves, this amounts to introducing the appropriate wave number, ω^2/g, and rewriting equation (1.10)

$$r(X, Y, t) = \int_0^\infty \cos\left[\frac{\omega^2}{g}(X\cos\chi + Y\sin\chi) - \omega t - \epsilon(\omega)\right]\cdot\sqrt{[r(\omega)]^2\,d\omega} \tag{1.17}$$

At the origin this reduces to equation (1.10). The term involving the coordinates of any point X, Y is a constant for given values of ω and χ.

The factor representing the variation with distance can then be absorbed in the random phase $\epsilon(\omega)$. With this change equation (1.17) becomes

$$r(X, Y, t) = \int_0^\infty \cos\left\{\omega t + \left[\epsilon(\omega) - \frac{\omega^2}{g}(X\cos\chi + Y\sin\chi)\right]\right\}\cdot\sqrt{[r(\omega)]^2\,d\omega} \tag{1.18}$$

This expression can also be represented as a partial sum taken over the same net as for equation (1.15).

$$r(X, Y, t) = \sum_{n=0}^{q}\cos\left\{\omega_{2n+1}t + \epsilon(\omega_{2n+1}) - \frac{(\omega_{2n+1})^2}{g}[X\cos\chi + Y\sin\chi]\right\}\sqrt{[r(\omega_{2n+1})]^2\,(\omega_{2n+2} - \omega_{2n})} \tag{1.19}$$

An important conclusion can now be brought out. For any fixed values of X and Y, say X_1 and Y_1, it is always possible to add (or subtract) an integral number of 2π's to the random phase

$$\epsilon(\omega_{2n+1}) - \frac{(\omega_{2n+1})^2}{g}[X_1 \cos\chi + Y_1 \sin\chi].$$

In this manner a new random phase is obtained satisfying the equation

$$0 \leqq \epsilon(\omega_{2n+1}) - \frac{(\omega_{2n+1})^2}{g}[X_1 \cos\chi + Y_1 \sin\chi] + 2\pi N = \epsilon'(\omega_{2n+1}) \leqq 2\pi. \quad (1.20)$$

This new random phase will be distributed according to the same probability law that governs the distribution of the original $\epsilon(\omega_{2n+1})$. Consequently, $\epsilon'(\omega_{2n+1})$ is another point set function like $\epsilon(\omega_{2n+1})$. A Gaussian distribution at the origin is, therefore, reproduced as a Gaussian distribution over the extended surface of the sea. The result is that the energy spectrum is the same at any fixed point X_1, Y_1. In a statistical sense, then, the sea surface has the same properties at all points.

(2) *Short-Crested Waves.* The extension to short-crested waves is carried out by the simple superposition of long-crested systems differing in direction of travel χ. With χ introduced as a variable, equation (1.17) becomes

$$r(X, Y, t) = \int_{-\pi}^{\pi}\int_0^{\infty} \cos\left[\frac{\omega^2}{g}(X\cos\chi + Y\sin\chi) - \omega t - \epsilon(\omega, \chi)\right] \cdot \sqrt{[r(\omega, \chi)]^2\, d\omega\, d\chi} \quad (1.21)$$

and the representation by a partial sum is now

$$r(X, Y, t) = \sum_{m=0}^{m=p}\sum_{n=0}^{n=q} \cos\left[\omega_{2n+1}t + \epsilon(\omega_{2n+1}, \chi_{2m+1}) - \frac{(\omega_{2n+1})^2}{g}(X\cos\chi_{2m+1} + Y\sin\chi_{2m+1})\right] \cdot \sqrt{[r(\omega_{2n+1}, \chi_{2m+1})]^2(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})} \quad (1.22)$$

The integral of expression (1.17) is obtained as the limit of a sum as follows:

(a) Divide the range of integration ($\omega - \chi$ polar coordinate plane) by a net formed by dividing the modulus by a series of arbitrarily spaced points: $0, \omega_1, \omega_2 \ldots \omega_{2q}$ and the argument by a series of arbitrarily spaced angles $-\pi, \chi_1, \chi_2 \ldots \pi$.

(b) Obtain $[r(\omega, \chi)]^2$ for all alternate points $(2n + 1, 2m + 1)$.

(c) Obtain the element $(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})$.

(d) Calculate $\sqrt{[r(\omega_{2n+1}, \chi_{2m+1})]^2(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})}$.

(e) Select $\epsilon(\omega, \chi)$ at random for each point $(\omega_{2n+1}, \chi_{2m+1})$ obtaining $\epsilon(\omega_{2n+1}, \chi_{2m+1})$ and calculate

$$\cos\left[\omega_{2n+1}t + \epsilon(\omega_{2n+1}, \chi_{2m+1}) - \frac{(\omega_{2n+1})^2}{g}(X\cos\chi_{2m+1} + Y\sin\chi_{2m+1})\right]$$

In the random selection of $\epsilon(\omega, \chi)$ the same law holds as expressed in equation (1.13) except that now ϵ is a function of both ω and χ.

(f) Form the partial sum

$$\sum_{m=0}^{p}\sum_{n=0}^{q} \cos\left[\omega_{2n+1}t + \epsilon(\omega_{2n+1}, \chi_{2m+1}) - \frac{(\omega_{2n+1})^2}{g}(X\cos\chi_{2m+1} + Y\sin\chi_{2m+1})\right] \cdot \sqrt{[r(\omega_{2n+1}, \chi_{2m+1})]^2(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})}$$

Then the limit of the partial sum given by equation (1.22) as ω_{2q+2} and p approach infinity, and as $(\omega_{2n+2} - \omega_{2n})$ approaches zero is the definition of the integral given by equation (1.21).

According to this representation the seaway is made up of an infinite number (in actual practice, of a large number) of wave systems varying both in frequency ω and in direction of travel χ. These systems are of infinitesimal (in actual practice, of very small) amplitude given by $\sqrt{[r(\omega, \chi)]^2\, d\omega\, d\chi}$ or, in actual application, by

$$\sqrt{[r(\omega_{2n+1}, \chi_{2m+1})]^2(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})}$$

and each wave system has a corresponding phase lag $\epsilon(\omega, \chi)$, in actual application $\epsilon(\omega_{2n+1}, \chi_{2m+1})$, chosen at random.

The representation of the short-crested seaway as a function of space and time reduces to the representation of the seaway as a function of time given by equation (1.10) whenever the height of

the waves is observed at a fixed point. The energy spectrum as a function of ω and χ is related to the energy spectrum as a function of ω alone by the relation

$$[r(\omega)]^2 = \int_{-\pi}^{\pi} [r(\omega, \chi)]^2 \, d\chi \qquad (1.23)$$

A proof of this statement is outlined as follows: Consider a partial sum as given by equation (1.22) with m quite large and consider also a summation over m for a fixed value of n. Since all terms have the same frequency, their sum is equal to one simple sine wave whose frequency is the same as that of the terms and whose amplitude is determined by the separate amplitudes and random phases of all the terms in the sum. This amplitude can be shown to have a value picked at random from the probability distribution function which is given by

$$p(k_1 < a_{2n+1} < k_2) = \int_{k_1}^{k_2} \frac{2ke^{-k^2/\Delta R}}{\Delta R} \cdot dk \qquad (1.24)$$

where $k_1 \geq 0$ and

$$\Delta R = \int_{\omega_{2n}}^{\omega_{2n+2}} \int_{-\pi}^{\pi} [r(\omega, \chi)]^2 \, d\chi \qquad (1.25)$$

The result is that if $r(t)$ is now summed over n it can be represented schematically by

$$r(t) = \sum_{n=0}^{\infty} a_{2n+1} \cos [\omega_{2n+1}t + \epsilon(\omega_{2n+1})] \qquad (1.26)$$

If the mesh of the net is made small it can be shown that over any small finite range of ω, say from ω_{2s} to ω_{2s+2j}, the sum of the squares of the amplitudes in equation (1.26) can be made negligibly different from the area under the frequency spectrum over the same range of ω. This is stated as follows

$$\lim_{j \to \infty} \sum_{n=s}^{s+j-1} (a_{2n+1})^2 = \int_{\omega_{2s}}^{\omega_{2s+2j}} [r(\omega)]^2 d\omega \qquad (1.27)$$

where the difference $(\omega_{2s+2j} - \omega_{2s})$ is kept constant as j approaches infinity.

The fact that wave records taken at different points all have the same theoretical frequency spectrum is a property of the short-crested Gaussian sea surface. Such a property is not exhibited by any short-crested model of the sea surface which consists of only a finite sum of sine waves.

Properties of the Seaway

Representation of the seaway by this extended form of the energy integral succeeds in bringing out certain of its properties which, although observed in nature, are not exhibited by other mathematical models. However, even this representation fails to indicate properties which are derivable from the non-linear features of the original boundary condition for the free surface. The following properties of the seaway, then, are indicated by the mathematical model advanced herein:

(a) The waves of the sea are short crested—in nature waves are always short crested. It is a matter of common observation that it is not possible to follow a given crest very far on the surface of the ocean in a direction along the crests.

(b) The sea surface is irregular and never repeats itself—this is the most characteristic feature of the sea.

(c) Wave crest forms are not conservative and as they travel along they tend to disappear. It is commonly observed that new crests are continuously formed and existing crests continuously destroyed.

In addition to these aspects, the mathematical model of this text explains the basic difference between a sea and a swell. The energy spectrum of a sea covers a wide range of significant values of ω and χ. The range of ω corresponds to periods extending from 1 to 30 seconds and the range of χ is in excess of $\pi/2$. An example of the energy spectrum for a sea is given in Fig. 1.6. As against this, the energy spectrum of a swell covers only

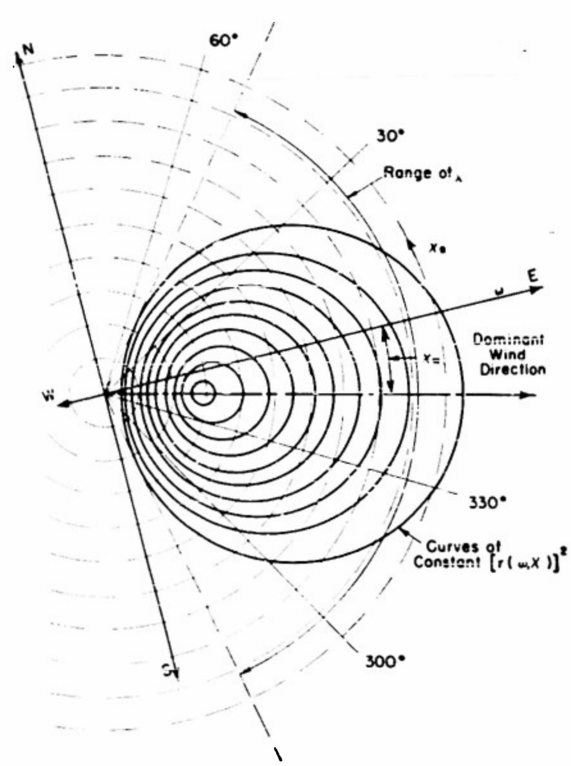


Fig. 1.6.—Energy Spectrum for a Sea (Idealized)

narrow ranges of ω and χ, the latter often smaller than 10 degrees. An example of an energy spectrum for a swell is given in Fig. 1.7. The wave systems making up a swell have all nearly the same direction and the same frequencies. In contrast to a sea which is always short crested and irregular, a swell is fairly long crested and exhibits a visible pattern. The crests of a swell can be observed to travel farther before they too die down and other crests form.

It appears to be a tenable conclusion, therefore, that the representation of the sea surface by the energy integral for the Gaussian case provides the most satisfactory mathematical model for describing the sea.

The Spectrum of the Seaway

There are two procedures for determining the spectrum of the seaway.

The first procedure requires the simultaneous recording of (a) the height of the surface as a function of time at a fixed point, and (b) the wave pattern from a considerable altitude. The former requires wave recorders that will give reliable results in the open sea. The latter consists in taking stereo-photographs by plane. Based upon the availability of such data, the technique to be followed to obtain the energy spectrum of the seaway has been discussed by Marks [11]. The energy spectrum as a function of wave frequency alone is not sufficient to describe the seaway completely.

It should be noted that the work of numerical analysis by this procedure requires an exorbitant amount of time unless electronic computers are available. This procedure is being pursued at present by the Woods Hole Oceanographic Institution.

The second procedure is to derive theoretically the spectrum of the seaway for certain idealized wind conditions. This work has been carried out by Neumann [12]. His theoretical spectrum exhibits many observed wave properties. Nevertheless, it still needs to be verified by observations as described above.

The Neumann spectrum, Fig. 1.8, applies to the state of the sea at a fixed point and is a function of the wave frequency alone. It is applied to the state of the sea over an area by introducing an angle χ_w measured from the wind direction.

$$[r(\omega, \chi_w)]^2 = \begin{cases} \dfrac{c}{\omega^6} e^{-2g^2/U^2\omega^2} \cos^2 \chi_w \\ \qquad \text{if } -\dfrac{\pi}{2} < \chi_w < \dfrac{\pi}{2} \\ 0 \text{ otherwise} \end{cases} \tag{1.28}$$

The constant $c = 3.05 \times 10^4$ cm^2 sec^{-5} or 32.9 ft^2 sec^{-5} and U is the wind velocity in cm/sec or ft/sec depending on the units of g. The angle χ_w is needed to explain the short crestedness of the sea surface, its variation as the square of the cosine is based on theoretical results and observations by Arthur [1].

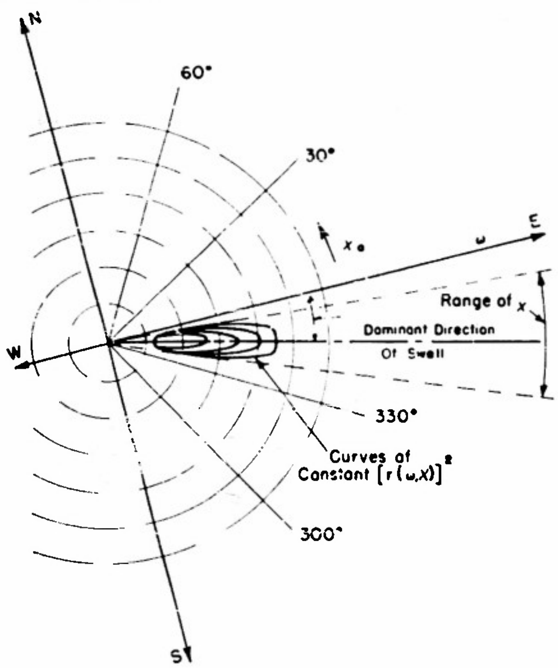


Fig. 1.7.—Energy Spectrum for a Swell (Idealized)

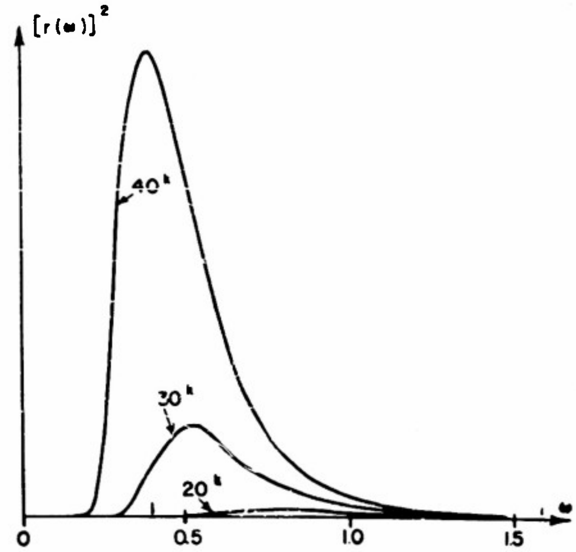


Fig. 1.8.—Neumann Spectrum of Wave Energy for Fully Arisen Sea at Wind Speeds of 20, 30 and 40 Knots

Note the displacement of the optimum band (maximum of spectral energy) from higher to lower frequencies with increasing wind speed.

TABLE 1.2.—MINIMUM FETCH AND MINIMUM DURATION NEEDED TO GENERATE A FULLY DEVELOPED SEA FOR VARIOUS WIND VELOCITIES

(V is in knots, F_m is in nautical miles and t_h is in hours)

V	F_m	t_h	V	F_m	t_h
10	10	2.4	34	420	30
12	18	3.8	36	500	34
14	28	5.2	38	600	38
16	40	6.6	40	710	42
18	55	8.3	42	830	47
20	75	10	44	960	52
22	100	12	46	1100	57
24	130	14	48	1250	63
26	180	17	50	1420	69
28	230	20	52	1610	75
30	280	23	54	1800	81
32	340	27	56	2100	88

Such a spectrum is obtained only if the wind has blown a sufficiently long time over a great enough fetch, the time and fetch required to develop this spectrum being functions of the wind velocity, see Table 1.2. From this table it is possible to conclude that conditions in nature will usually permit the waves of the sea to attain their fully developed state for winds up to 32 knots and perhaps of somewhat greater velocities. But the long durations and boundless fetches necessary for waves to attain their fully developed state under the action of strong winds are seldom obtained in nature. To illustrate, a fully developed sea would result if a wind of 52 knots blew for 80 hours over a fetch of 1800 nautical miles. Such conditions are not common.

If the sea is in a fully developed state the integration of equation (1.28) results in a formula for the cumulative energy density, R, which is proportional to the fifth power of the wind velocity.

$$R(\mathrm{m}^2) = 0.622 \left(\frac{U}{10}\right)^5 \ (\mathrm{m/sec})$$

$$R(\mathrm{ft}^2) = 0.242 \left(\frac{U}{10}\right)^5 \ (\mathrm{knots}) \qquad (1.29)$$

According to the results of Rice, the envelope of the wave state at a point, equation (1:10) has a distribution, known in statistics as the target or chi distribution, given by

$$P[0 < E(t_1) < M] = \int_0^M \frac{2\xi}{R} e^{-\xi^2/R}\, d\xi \quad (1.30)$$

where $E(t)$ is the envelope of the wave record, Fig. 1.9. The expression is read as follows: The probability that at a time t_1 chosen at random, the envelope of the record of the sea surface at a fixed point will be less than M is given by the integral from zero to M of the target distribution.

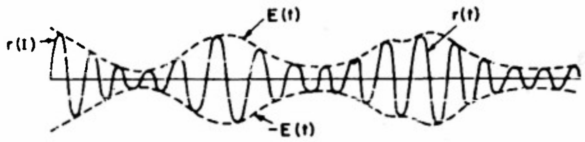


FIG. 1.9.—ENVELOPE OF WAVE RECORD

For a seaway of narrow spectrum the envelope and the wave amplitudes approach each other and they can be used interchangeably. However, the spectrum of the seaway is not generally narrow. Yet even in this case such a distribution can be used as a basis of approximating the average wave height. This average wave height (crest to trough) results from the first moment about the origin of the distribution and is given by

$$\bar{h} = 1.77\sqrt{R} \qquad (1.31)$$

A wave height parameter frequently used is the significant height. This is defined to be the average height of the one-third highest waves. This can be obtained also from the above probability distribution by considering only that portion of the area under the high end of the probability distribution function which is equal to one-third of the total area. The resulting significant height is

$$\bar{h}_{1/3} = 2.83\sqrt{R} \qquad (1.32)$$

Similarly the average height of the one-tenth highest waves to pass a given point of observation can be found. This amounts to

$$\bar{h}_{1/10} = 3.60\sqrt{R} \qquad (1.33)$$

The frequency at which the spectrum is a maximum is obtained by differentiating equation (1.28) and is

$$\omega_m = \sqrt{\frac{2}{3}} \cdot \frac{g}{U} = \frac{g}{1.22U} \qquad (1.34)$$

Since the wave velocity $= g/\omega$ some interesting results follow:

(a) The spectral frequencies at which most of the energy is concentrated are of rather low values.

(b) The speed of the dominant wave component is 1.22 times that of the wind that generated the seaway.

(c) Those wave components carrying the most energy travel with a phase velocity faster than the generating wind.

As the frequency is decreased from that at which the spectrum attains its maximum value, the energy present rapidly falls off to very small values. The slope of the spectrum is a maximum at a frequency known as the cut-off frequency. This is given by

$$\omega_c = \sqrt{\frac{8}{15+\sqrt{57}}}\cdot\frac{g}{U} = \frac{g}{1.68U} \qquad (1.35)$$

Thus the ratio of wave-crest speed to wind speed is still not equal to two when the spectrum is falling off extremely rapidly. In fact, there is practically no energy at all in the storm-generated seaway which is associated with spectral frequencies whose group velocity (equal to one-half wave crest velocity) exceeds that of the velocity of the wind in the generating area.

A seaway is often defined by an average wave period or frequency. The average wave period is defined as twice the average time interval between successive zeros in a wave record. The average wave frequency is the reciprocal thereof. The procedure for obtaining such average values was given by Rice. When applied to the Neumann spectrum the average frequency obtained is equal to

$$\bar{\omega} = \frac{2}{\sqrt{3}}\cdot\frac{g}{U} = \frac{g}{0.866U} \qquad (1.36)$$

This average frequency is seen to be much larger than the dominant spectral frequency ω_m. Consequently, the average time interval between the crests as observed in a fully developed seaway at a fixed point by stop watch will give an average period which is too small and which will not reflect the part of the spectrum at which most of the energy is concentrated.

This is not too difficult to explain intuitively. It is evident that more waves with a five-second period can pass a given point during a given time of observation than can waves with a ten second period. The average thus tends to be lower due to the overemphasis placed upon the shorter periods.

As soon as the waves spread out from the area in which they were generated the frequencies disperse. Waves are then observed to pass a point at a distance from the generating area according to an ascending order of wave frequency, since their group velocity is given by $g/2\omega$. Thus the current belief that the swell increases in period as it travels away from the storm is explained theoretically. The highest swell is observed to have the period associated with ω_m.

The foregoing has an important consequence in the study of ship motions. If an average stopwatch period is used to replace a spectrum, misleading results can be obtained because the observed period would be too low to explain the behavior of the ship. This would be especially serious if the ship's resonant frequency corresponded to ω_m and not to $\bar{\omega}$. It can be concluded that the only way in which to explain correctly ship motions is to relate them to the properties of the wave energy spectrum.

The spectrum discussed above corresponded to a fully developed sea. When the wind begins to blow over a calm stretch of water, waves begin to form and to grow in height with time. As shown by Neumann, the spectrum develops from the high frequency end. For any given duration and fetch, the wave spectrum is a portion of the fully developed spectrum. It has the property that all frequencies less than a certain value are simply missing. The lowest existing frequency is a predictable function of fetch and wind velocity and duration. It is therefore possible to obtain theoretically a first approximation to the spectrum of the seaway at any point and at any time. The procedure to be followed is given by Pierson, Neumann and James [15].

THE RESPONSE AMPLITUDE OPERATORS

The response amplitude operators are functions which yield the amplitude of the oscillatory response of a vessel along or about one of her principal axes when she is in a regular seaway of unit amplitude. There is thus a separate response amplitude operator for each degree of freedom.

Not all the oscillatory motions of a ship will be considered in this paper. Attention will be restricted to heave, pitch and roll, these being the most important oscillations to which a ship is subjected. Fortunately, these oscillations are also somewhat amenable to mathematical treatment. In contrast to surge, sway and yaw, these motions are of a pure oscillatory nature because of the ever-presence of a hydrostatic restoring force when the vessel is disturbed from her position of equilibrium.

In what follows, the assumption is made that surge, sway and yaw are negligible.

The response amplitude operators can be obtained either theoretically or experimentally. Empirical formulations of reasonable accuracy for most engineering applications can be developed by application of either method. The basis of both the theoretical and the experimental methods will now be briefly discussed.

Theory

It is convenient to recall, for ease of discussion, the essential features of an analytic solution of the differential equation defining those ship motions considered herein. Such a solution for a vessel operating in a regular sea has been considered in two previous papers [24] [20]. Because the analytic solution rests upon several powerful assumptions or restrictions and because these act to limit the applicability of the results of this paper also, they will be briefly stated here.

(a) Restriction of the theory to vessels of the displacement type. All other types of vessels, such as planing and hydrofoil craft, for which the sustention is developed dynamically, are excluded.

(b) Acceptance of the Froude-Krylov hypothesis (the waves act on the ship, but the ship does not act on the waves). This results in the neglect of all dynamic effects except those inherent in the structure of the free waves themselves. The seriousness of this assumption is as yet undetermined. It is evident, however, that its severity increases with a vessel's fullness (or, better, bluntness) of form.

(c) Restriction to uncoupled motions. Cross-coupling between heave, pitch and roll is assumed not to exist. Thus these motions are considered to be entirely independent of each other. The effect of this restriction is to limit the applicability of the theory only to vessels whose waterplanes are quasi-symmetric fore and aft.

(d) Restriction to linear aspects. The response is assumed to be a linear function of the exciting, restoring, damping and inertia forces. The effect of this restriction is:

(1) To limit consideration to wall-sided vessels, thus disregarding the effect of transverse curvature (flare, tumble-home, etc.).

(2) To assume that damping arises solely from wave generation. The error introduced thereby is small in heave and pitch. In roll, however, the largest component of damping arises from eddy-making (bilge-keels) and does not follow a linear law.

(3) To assume that the virtual mass and moment of inertia of a vessel are unaffected by speed of advance and frequency of oscillation.

As outlined in the Appendix, the differential equation of motion derived on the basis of these assumptions and restrictions is of the oscillatory type. In its simplest reduced form it is written as

$$\frac{d^2s}{dt^2} + 2h_s\frac{ds}{dt} + \nu_s^2 s = f_s \begin{pmatrix}\sin\\ \cos\end{pmatrix}(\omega_e t - \epsilon) \quad (2.1)$$

corresponding to equations (113) and (193) of reference [20]. Here s represents the response, i.e., a motion, either heave (z), pitch (ψ) or roll (φ), $2h_s$ represents a damping factor and ν_s is the natural undamped frequency of oscillation of the vessel. The excitation, f_s, is a complex expression involving both the wave characteristics and the form of the vessel. When the right-hand term is given as a convenient and simple periodic function of time, the following steady-state solutions obtain for the motions considered.

$$s(t) = E_s \cdot \mu_s \cdot r_m \begin{pmatrix}\sin\\ \cos\end{pmatrix}(\omega_e t - \epsilon_s) \quad (2.2)$$

Here the subscript s denotes that the terms to which applied are to be evaluated for a particular

motion. The subscript becomes z for heave, φ for roll and ψ for pitch. E_s is a force factor of complex expression which, like f_s, depends on the form of the vessel and the wave characteristics. μ_s is a magnification factor defined below. r_m is the surface wave amplitude. ϵ_s is a phase lag dependent upon the form of the vessel and her damping characteristics.

The magnification factor is defined as

$$\mu_s = \frac{\nu_s^2}{\sqrt{(\nu_s^2 - \omega_e^2)^2 + (2h_s\omega_e)^2}} = \frac{1}{\sqrt{[1 - (\Lambda_s)^2_e]^2 + \kappa_s^2(\Lambda_s)_e^2}} \tag{2.3}$$

where $(\Lambda_s)_e$, the effective tuning ratio $\triangleq \omega_e/\nu_s$.

For a surface wave amplitude of unity ($r_m = 1$) the amplitude of the response is defined as the response amplitude operator.

$$A_s \triangleq E_s \cdot \mu_s \tag{2.4}$$

Theoretical expressions for the response amplitude operators in heave (A_z), pitch (A_ψ) and roll (A_φ) are derived in the Appendix.

These operators are obtained as the product of two component factors: a force factor, E_s, and a magnification factor, μ_s. For a given vessel the force factors are basically functions of the length of waves in which the vessel finds herself and of the wave heading relative to the ship. Because of the interrelation between wave frequency and wave length, the former is substituted for the latter and the force factor is simply written as a function of the natural wave frequency and wave heading, $E_s(\omega, \chi)$. The force factor is thus related to the relative coordinate system.

For a given vessel the magnification factor depends solely on the frequency of wave encounter, ω_e, and may be written as $\mu_s(\omega_e)$. The magnification factor is consequently related to the vessel's coordinate system.

It is obviously inconvenient to work with a function made up of two factors, one of which is related to a fixed, the other to a moving, system of coordinates. Therefore, it is first of all necessary that both factors be expressed in terms of the coordinates of a same system. The relative coordinate system is chosen for this purpose. This choice makes for an inconvenience in expressing μ_s; but this inconvenience is appreciably less than that due to the clumsiness attendant in subsequent derivations upon expressing E_s in terms of ω_e.

Thus, the response amplitude operators will first be written in terms of ω, χ and v, the ship's velocity.

To the end of relating the magnification factor to the relative coordinate system, it is necessary to introduce the frequency of wave encounter. This is obtained by substituting in the expression $\omega = g/c$ obtained from equation (1.2) the relative wave celerity in lieu of the absolute. Since the component of ship's velocity in direction of wave travel is $v \cos \chi$, the relative wave celerity is $c - v \cos \chi \triangleq c(1 - \alpha)$ where

$$\alpha \triangleq \frac{v \cos \chi}{c} \tag{2.5}$$

Then

$$\omega_e = \omega(1 - \alpha) = \omega - \frac{\omega^2}{g} \cdot v \cos \chi \tag{2.6}$$

and

$$(\Lambda_e)_s = (1 - \alpha)\Lambda_s \tag{2.7}$$

where $\Lambda_s = \omega/\nu_s$ is now based on the wave frequency instead of on the frequency of encounter.

The magnification factor is then written as

$$\mu_s = \frac{1}{\sqrt{[1 - (1 - \alpha)^2\Lambda_s^2]^2 + (1 - \alpha)^2\kappa_s^2\Lambda_s^2}} \tag{2.8}$$

Its well-known properties are illustrated in Fig. 2.1.

With this expression for the magnification factor, the amplitude response operators are written with reference to a coordinate system fixed in space. It now becomes necessary to relate them to the moving vessel. This transformation will be discussed in the section on the frequency mapping.

Experiment

The experimental is an alternative approach to the theoretical for obtaining the response amplitude operators. Should this be the only information desired, the experimentation is relatively simple: It consists in running a model at various speeds in a train of uniform waves while simultaneous recordings are made of the motions and wave profiles. The response amplitude operators are obtained directly by relating the amplitude of each motion to the wave amplitude. The experimentation is carried out over a wide range of wave frequencies and directions for each model speed of interest.

Up to the present it has been possible to obtain response amplitude operators only for severely restricted conditions. In heave and pitch the conditions of experimentation have been limited to head and following seas; in roll, to beam seas and zero speed of advance. This obviously unsatisfactory situation is being remedied at present through the setting up of a facility at the Colo-

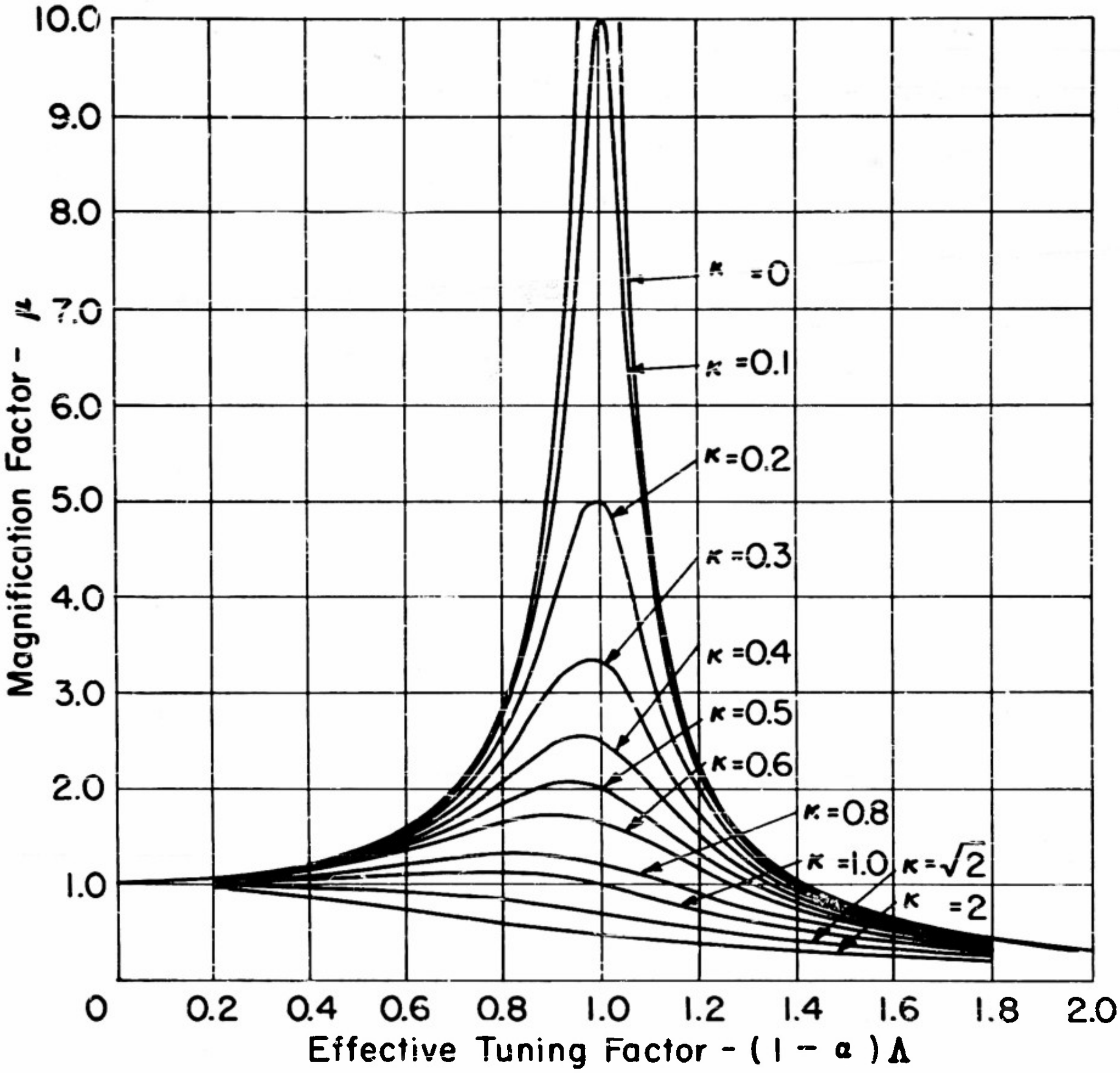


FIG. 2.1.—MAGNIFICATION FACTOR

rado Agricultural and Mechanical College that will permit experimentation in oblique seas.

If in addition to the response amplitude operators it is desired to obtain information on the inertia and damping coefficients, on the force and magnification factors and on the validity of the assumptions underlying the basic theory, then the above simple tests in waves must be supplemented by more elaborate tests in which the model is simultaneously towed in calm water and acted upon by forces imposed by an oscillator.

In order to connect theory and experimentation, a brief outline will be given of the principles involved.

Heave. Consider first the heaving motion. The experimental set-up for this case is illustrated in Fig 2.2. The oscillator in this case consists of a motor acting through an eccentric, a shaft and the sensing element of a dynamometer to impose upon the model at its center of gravity a sinusoidally varying force equal to $F_z \cos \omega_e t$. The model is towed in calm water. The equation of the oscillatory motion of the system is given by

$$M_z \frac{d^2z}{dt^2} + N_z \frac{dz}{dt} + R_z z = F_z \cos \omega_e t \quad (2.9)$$

where

M_z = the virtual mass of the system
N_z = a coefficient of linear damping
R_z = the restoration coefficient
F_z = the amplitude of the excitation

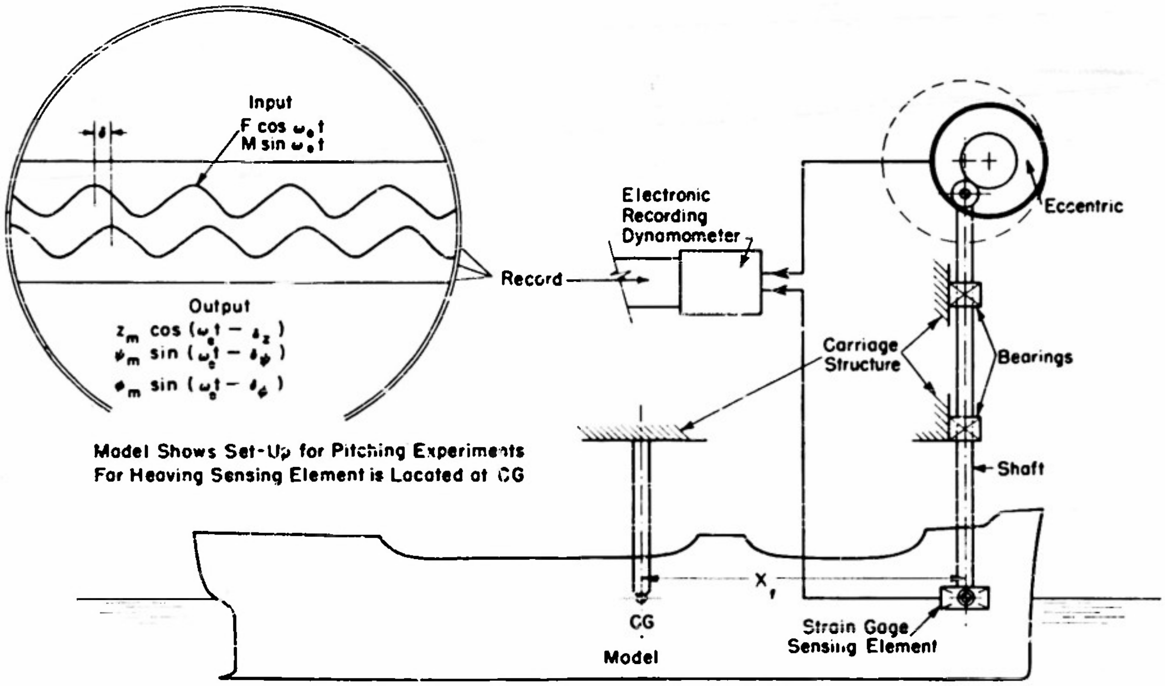


FIG. 2.2.—EXPERIMENTAL SET-UP WITH OSCILLATOR TO OBTAIN RESPONSE AMPLITUDE OPERATORS

For a fuller exposition of these terms the reader is referred to the Appendix.

The following is known of the factors of the equation:

ω_e is controlled and therefore known.
$F_z \cos \omega_e t$ and $z(t)$ are recorded.
δ_z, the phase lag between exciting force and response is determined from the record.
R_z is a known function of the model's geometry. $R_z = \rho g A$, where A is the waterplane area.
M_z and N_z are to be derived.

If z can be written as

$$z = z_m \cos (\omega_e t - \delta_z) \tag{2.10}$$

a matter that can be readily verified, M_z and N_z are immediately derived as functions of ω_e

$$M_z(\omega_e) = \frac{1}{\omega_e^2}\left(R_z - \frac{F_z}{z_m}\cos \delta_z\right) \tag{2.11}$$

$$N_z(\omega_e) = \frac{F_z \sin \delta_z}{\omega_e z_m} \tag{2.12}$$

The inertia and dimensionless damping coefficients in heave are readily obtained from these equations. The first is

$$k_z(\omega_e) = \frac{M_z - \rho\nabla}{\rho\nabla} \tag{2.13}$$

and the second

$$\kappa_z(\omega_e) = \frac{N_z}{M_z \nu_z} \tag{2.14}$$

The natural frequency of heave, ν_z, is obtained by removing the excitation and allowing the model to oscillate freely after being disturbed from the position of equilibrium.

If now the model is removed from the oscillator and towed in a wave train, the equation of the system's oscillatory motion becomes

$$\begin{aligned} M_z \frac{d^2z}{dt^2} + N_z \frac{dz}{dt} + R_z z &= F_z \cos (\omega_e t - \epsilon_z) \\ &= f_z M_z \cos (\omega_e t - \epsilon_z) \\ &= \nu_z^2 r_m E_z M_z \cos (\omega_e t - \epsilon_z) \\ &= r_m R_z E_z \cos (\omega_e t - \epsilon_z) \end{aligned} \tag{2.15}$$

since $\nu_z^2 = R_z/M_z$. The phase lag ϵ_z is introduced not only by longitudinal asymmetry of form but also by dynamic effects which are not accounted for in the theory based upon the Froude-Krylov hypothesis. It therefore can be written as the sum of two components $\epsilon_z = (\epsilon_1)_z + (\epsilon_2)_z$. The first component can be calculated. It has the expression

$$(\epsilon_1)_z = arctg \frac{G_z^a}{G_z^s} \tag{2.16}$$

where G_z^a and G_z^s are, respectively, the antisymmetric and the symmetric exciting functions.

They are given by equations (a-90) and (a-93) in the Appendix. For a longitudinally symmetric model $(\epsilon_1)_z$ is zero. The second component, $(\epsilon_2)_z$, can be obtained only experimentally. This is done in the manner indicated below.

If the solution to equation (2.15) is assumed to be of the form

$$z = z_m \cos(\omega_e t - \epsilon_z - \delta_z) \qquad (2.17)$$

the force factor E_z and the phase lag δ_z are readily derived as functions of ω_e

$$E_z(\omega_e) = \frac{z_m}{R_z r_m}\sqrt{(R_z - \omega_e^2 M_z)^2 + \omega_e^2 N_z^2} \qquad (2.18)$$

and

$$\delta_z(\omega_e) = arctg\,\frac{\omega_e N_z}{R_z - \omega_e^2 M_z} \qquad (2.19)$$

The phase lag $(\epsilon_z + \delta_z)$ is the time increment from the instant the crest of a wave goes by the section at the center of mass to the instant of maximum heave. It can be obtained from synchronized heave and photographic records. Thus with $(\epsilon_z + \delta_z)$ known and δ_z calculated by the foregoing expression, ϵ_z is derived as a function of ω_e.

Pitch. The procedure for obtaining the corresponding data in pitch follows along parallel lines. The calm water experiments are carried out with the model free to rotate about a transverse axis through the center of gravity. The oscillator is made to apply the sinusoidally varying force F_ψ at a distance X_f forward (or aft) of this center. The applied moment is now

$$M_\psi \sin \omega_e t$$

where $M_\psi = F_\psi X_f$ and the equation of motion of the system becomes

$$I_\psi \frac{d^2\psi}{dt^2} + N_\psi \frac{d\psi}{dt} + R_\psi \psi = M_\psi \sin \omega_e t \qquad (2.20)$$

On the basis of an assumption as to the motion similar to that for heave, the virtual mass and damping coefficient are given a function of ω_e by

$$I_\psi(\omega_e) = \frac{1}{\omega_e^2}\left(R_\psi - \frac{M_\psi}{\psi}\cos\delta_\psi\right) \qquad (2.21)$$

$$N_\psi(\omega_e) = \frac{M_\psi \sin\delta_\psi}{\omega_e \psi} \qquad (2.22)$$

The force factor and phase lag obtained from the tests in waves are then expressed as a function of ω_e by

$$E_\psi(\omega_e) = \frac{\psi}{R_\psi k r_m}\sqrt{(R_\psi - \omega_e^2 I_\psi)^2 + \omega_e^2 N_\psi^2} \qquad (2.23)$$

$$\delta_\psi(\omega_e) = arctg\,\frac{\omega_e N_\psi}{R_\psi - \omega_e^2 I_\psi} \qquad (2.24)$$

Roll. The hydrodynamic coefficients and the force factor and phase lag in roll are obtained in a manner almost identical to that for pitch. In the calm water experiments the model is free to rotate about a longitudinal axis through the center of gravity. The sinusoidally varying force F_φ is now applied at a distance Y_f from the centerline and in a transverse plane through the center of gravity. The applied moment now equaling $M_\varphi \sin \omega_e t$ where $M_\varphi = F_\varphi \cdot Y_f$, the equation of motion of the system is written

$$I_\varphi \frac{d^2\varphi}{dt^2} + N_\varphi \frac{d\varphi}{dt} + R_\varphi \varphi = M_\varphi \sin w_e t \qquad (2.25)$$

Again following the same assumption as for heave and pitch the virtual mass and damping coefficients are given by

$$I_\varphi(\omega_e) = \frac{1}{\omega_e^2}\left(R_\varphi - \frac{M_\varphi}{\varphi}\cos\delta_\varphi\right) \qquad (2.26)$$

$$N_\varphi(\omega_e) = \frac{M_\varphi \sin\delta_\varphi}{\omega_e \varphi} \qquad (2.27)$$

The tests at zero speed in waves coming from abeam result in force factors and phase lags expressible as

$$E_\varphi(\omega_e) = \frac{\varphi}{R_\varphi k r_m}\sqrt{(R_\varphi - \omega_e^2 I_\varphi)^2 + \omega_e^2 N_\varphi^2} \qquad (2.28)$$

$$\delta_\varphi(\omega_e) = arctg\,\frac{\omega_e N_\varphi}{R_\varphi - \omega_e^2 I_\varphi} \qquad (2.29)$$

THE FREQUENCY MAPPING

The previous section was a discussion of the amplitudes of the ship's responses to a regular seaway of unit amplitude. The amplitude response operators were derived with reference to the relative system of coordinates fixed in space and so oriented that the positive direction of the abscissa corresponded to the ship's heading. In this section a discussion will be given of the frequency response of the vessel to the same regular seaway.

The vessel responds to the seaway with the frequency of wave encounter, ω_e. This is the wave frequency modified by the ship's velocity and is derived by the transformation of coordinates from the relative to the vessel's system, see Table 1.1. If the seaway is described with reference to the latter system, equation (1.1) takes the following form.

$$r(X_e Y_e t) = r_m \cos\left[\frac{\omega^2}{g}(X_e \cos\chi + Y_e \sin\chi) - (\omega - \frac{\omega^2}{g}\cdot v\cos\chi)t + \epsilon\right] \qquad (3.1)$$

and the frequency of encounter, or effective frequency, is given by the multiplier of t in this equation. The wave length remains unaffected by the change in coordinate system.

The Frequency of Encounter

It is necessary to discuss somewhat more fully the frequency of encounter

$$\omega_e \triangleq \omega - \frac{\omega^2}{g}\cdot v\cos\chi \qquad (3.2)$$

Without loss of generality in what follows the ship is assumed to have a positive forward velocity.

As is evident from the foregoing expression, the frequency of encounter depends on the heading angle. For waves approaching the ship from the bow ($90° < \chi < 270°$), this frequency is always greater than the wave frequency. It is always less thereof for waves approaching from the stern ($0 < \chi < 90°$ and $270° < \chi < 360°$).

In the latter case, however, three distinct possibilities may arise depending on the relative velocity of ship and wave; more precisely, on the ratio α defined as

$$\alpha \triangleq \frac{v\cos\chi}{c} = \frac{\omega v\cos\chi}{g} \qquad (3.3)$$

where $v\cos\chi$ is the effective velocity of the ship, namely, the component of ship's velocity in the direction of wave travel. With this notation the frequency of encounter becomes simply[8]

$$\omega_e = \omega(1 - \alpha) \qquad (3.4)$$

(a) If $\alpha < 1$, the frequency of encounter is always positive and the waves overtake and pass the ship from astern.

(b) If $\alpha = 1$, the frequency of encounter is zero; hence the ship remains continuously in the same position relative to the wave profile.

(c) If $\alpha > 1$, the ship outraces the waves and their crests actually appear to be moving from the ship's bow toward her stern. In this case ω_e is negative.

The frequency of encounter attains its maximum value when

$$\frac{d\omega_e}{d\omega} = 1 - \frac{2\omega}{g}\cdot v\cos\chi = 0 \qquad (3.5)$$

i.e., when

$$\alpha = \frac{1}{2} \qquad (3.6)$$

It might be opportune to make some remarks on the significance of negative frequencies. It should be observed that only derived frequencies can be negative; observed frequencies are always positive. A negative frequency related to a moving system indicates a direction of motion opposite that observed. Two possible situations may arise depending upon whether the frequency of encounter is derived from the wave frequency or vice versa.

If the seaway is known by observation, both the wave frequency and direction, ω and χ, are known directly. The first is logically positive. The frequency of encounter is then obtained by derivation and may result either positive or negative. If negative, the apparent direction of wave propagation, χ_e, is opposite the real, i.e.,

$$\chi_e = \chi + \pi \qquad (3.7)$$

If, on the other hand, the ship motions are known by observation, then it is the frequency of

[8] Note that this notation is deceptively simple because actually α is a function of ω. It does, however, serve to keep the formulae in the text relatively concise.

wave encounter that is known directly; in this case it is always positive. The apparent direction of wave progress is also obtained by observation. The wave frequency is then obtained by derivation and may result either positive or negative. If negative, the real direction of wave travel is opposite the apparent, i.e.,

$$\chi = \chi_e - \pi \tag{3.8}$$

Thus in both cases a negative frequency (either wave or encounter) is equivalent to the corresponding positive frequency and a direction of wave travel (either real or apparent) opposite the observed one. Symbolically this may be expressed as follows:

$$(-\omega_e, \chi_e) \sim (+\omega_e, \chi_e - \pi)$$
$$(-\omega, \chi + \pi) \sim (+\omega, \chi) \tag{3.9}$$

Mathematically the foregoing equivalent expressions may be used interchangeably provided extreme care is taken in interpreting the ranges of integration and the signs of the terms which result in subsequent derivations. It is possible, however, to achieve a consistent system in which all frequencies are positive. This will be used herein.

Reference is now made to Fig. 3.1 which is a polar plot of equation (3.2). Here ω_e is shown as a function of ω and χ. The curves represent constant values of $\omega_e v/g$. Some of the characteristics exhibited by this plot are listed as follows:

(a) ω_e is a single-valued function of ω and χ.

(b) The lines α = constant represent vertical ordinates.

(c) All positive values of ω_e in terms of equation (3.2) are to be found to the left of $\alpha = 1$.

(d) All values of ω_e to the right of the vertical $\alpha = 1$ are negative in terms of equation (3.2) and all negative values of ω_e are asymptotic to this line.

(e) The curve for $\omega_e = g/4v$ crosses the reference line $\chi = 0$ at $\alpha = \frac{1}{2}$, and becomes asymptotic to the line $\alpha = 1$. As indicated above, at this value of α the effective frequency is zero.

(f) For all positive values $\omega_e < g/4v$ the curves of constant ω_e consist of two disconnected branches.

(1) The first branch is a closed loop around the origin.

(2) The second branch is a bow-shaped line to the left of $\alpha = 1$ and asymptotic thereto for large values of ω.

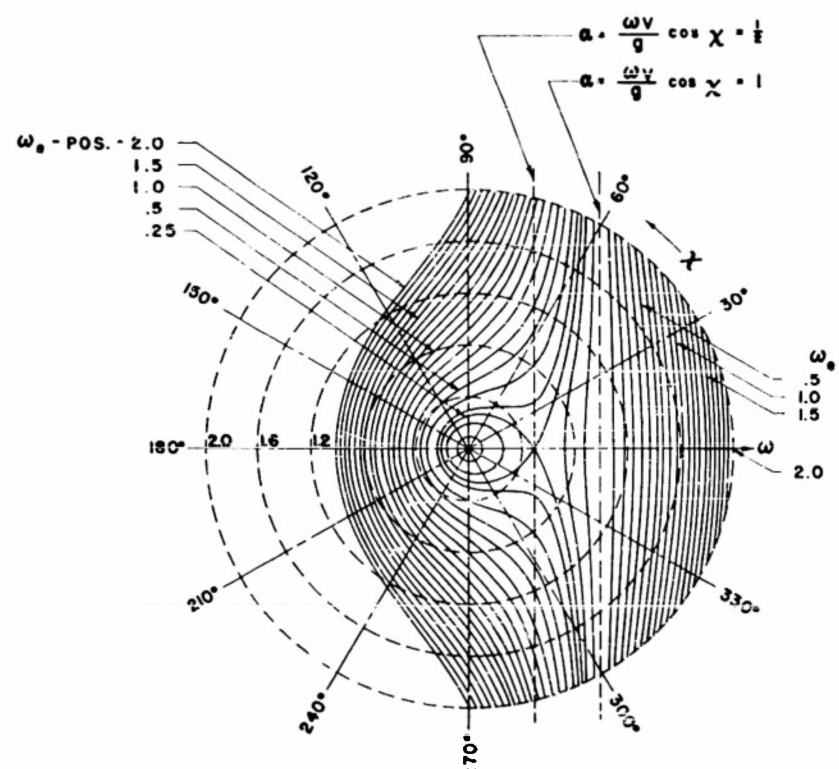


FIG. 3.1.—LINES OF CONSTANT ω_e IN THE ω-χ PLANE

Coordinates and curves are lines of constant $\omega \cdot v/g$ and $\omega_e \cdot v/g$. Thus to get true values multiply the scale values by g/v. In this figure, circles of constant ω have the values, $\omega \cdot v/g$ = 0.4, 0.8, 1.2, 1.6 and 2.0. The curves of constant ω_e are for constant increments of $\Delta \omega_e \cdot v/g$ = 0.10. In addition, the curve, $\omega_e \cdot v/g$ = 0.25, is given.

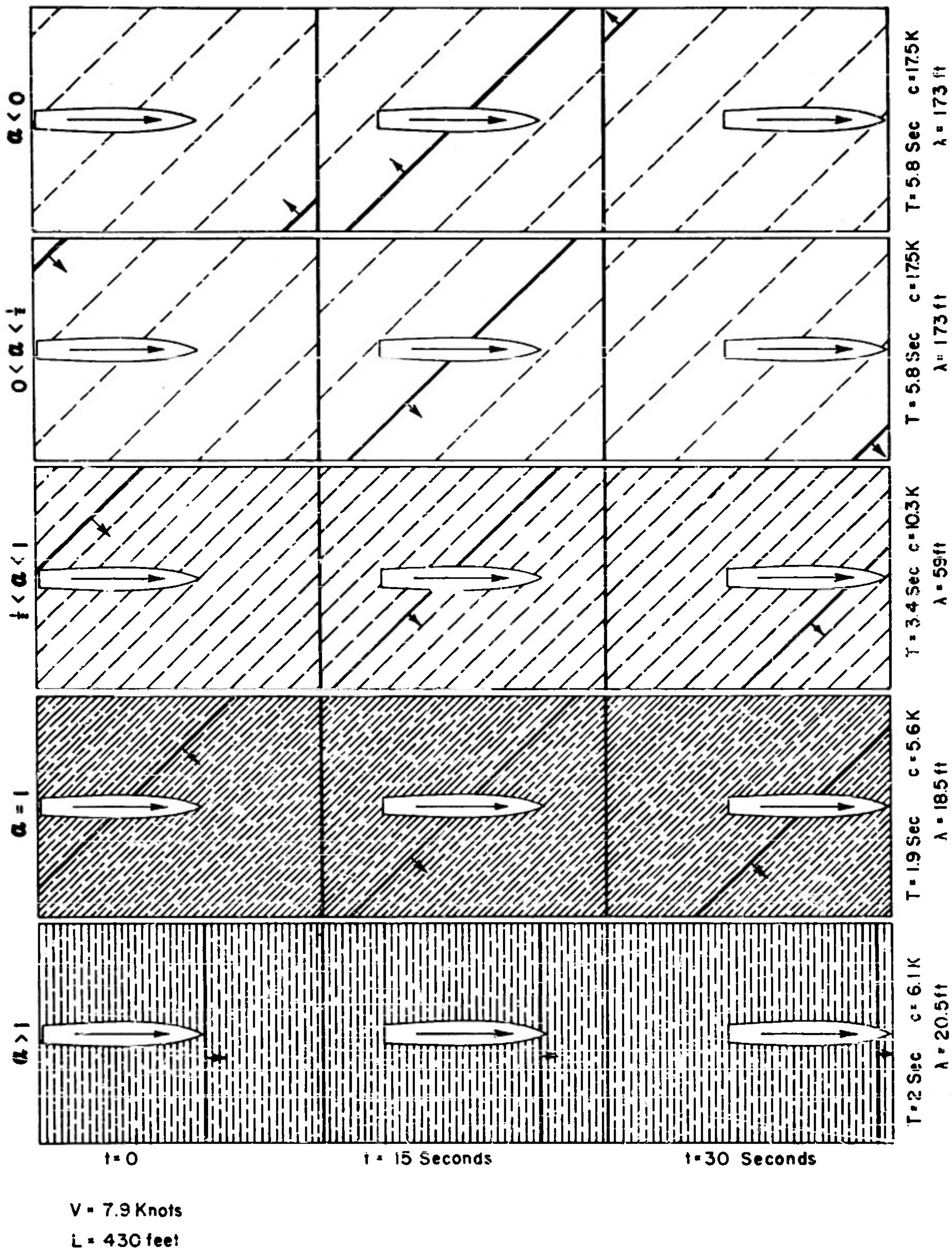


Fig. 3.2.—Schematic Interpretation of Fig. 3.1

Each rectangle is at the same fixed place in the Xa—Ya plane.

(g) The ratio of ship velocity, v, to wave velocity, g/ω, increases on going outwardly along a radial line χ = constant.

(h) If the ship has a velocity component in the direction of wave travel, $\cos\chi > 0$, ω_e equals zero at the origin, attains a maximum at $\alpha = \frac{1}{2}$, is zero again at $\alpha = 1$. Beyond this line it becomes negative decreasing indefinitely with increasing ω.

The schematic interpretation of Fig. 3.1 is given in Fig. 3.2. Four cases are possible:

(a) $\alpha < 0$: The waves approach the ship from the bow and ω_e is everywhere greater than ω.

(b) $0 < \alpha < \frac{1}{2}$: The waves approach the ship from the stern but they travel so rapidly that ω_e is only slightly less than ω.

(c) $\frac{1}{2} < \alpha < 1$: The waves still approach the ship from aft, but they travel so slowly compared to the ship that ω_e begins to decrease again.

(d) $\alpha > 1$: The ship outraces the waves and the waves appear to come from a direction opposite to that of their origin: ω_e is negative.

For purposes of consistency in the derivation which follows, all frequencies will be defined so as to be positive. To this end for the region of the ω-χ polar coordinate system where $\alpha > 1$, ω_e is redefined by

$$\omega_e \triangleq -\omega + \frac{\omega^2 v}{g}\cos\chi \qquad (3.10)$$

and χ_e by

$$\chi_e \triangleq \chi + \pi \qquad (3.11)$$

With ω_e defined by (3.10) and χ_e defined by (3.11) for $\alpha > 1$, and with ω_e defined by (3.2) and χ_e equal to χ for $\alpha < 1$, equation (3.1) becomes

$$r(X_e, Y_e, t) = r_m \cos\left[\frac{\omega^2}{g}(X_e\cos\chi_e + Y_e\sin\chi_e) - \omega_e t + \epsilon\right] \qquad (3.12)$$

everywhere in the ω_e-χ_e plane.

The Inverse Functions

Up to this point ω_e has been defined as a function of ω and χ. It is now necessary to obtain ω as a function of ω_e and χ_e. The inversion is made as follows:

If equation (3.2) is solved for ω one obtains

$$\omega = \frac{1 \pm \sqrt{1 - (4\omega_e v\cos\chi_e)/g}}{(2v\cos\chi_e)/g} \qquad (3.13)$$

or[9]

$$\omega = \left[\frac{1 \pm \sqrt{1 - 4\alpha_e}}{2\alpha_e}\right]\omega_e \qquad (3.14)$$

where the solutions are valid for

$$\alpha_e \triangleq \frac{\omega_e v\cos\chi_e}{g} < \frac{1}{4} \qquad (3.15)$$

since $\chi_e = \chi$.

Evidently ω is not a single-valued function of ω_e and χ_e. Because of the double sign before the radical, two solutions are obtained on each side of the straight line

$$\omega = \frac{1}{(2v\cos\chi)/g} \qquad (3.16)$$

which corresponds to $\alpha = \frac{1}{2}$.

For this value of α

$$\alpha_e = \omega_e/2\omega = \frac{1}{2}(1 - \alpha) = \frac{1}{4} \qquad (3.17)$$

Thus the two regions in the $\omega - \chi$ plane which are separated by the vertical at $\alpha = \frac{1}{2}$ transform into overlapping regions in the ω_e-χ_e plane. This makes it expedient to treat these regions separately when inverting them. To this end two primitive and two inverted regions may be outlined as follows:

Range of primitive region in ω-χ plane	Range of corresponding inverted region in ω_e-χ_e plane
$\alpha < \frac{1}{2}$	$\alpha_e < \frac{1}{4}$
$\alpha > \frac{1}{2}$	$\alpha_e < \frac{1}{4}$

However, the second primitive region can be further subdivided depending upon whether ω_e is positive or negative, or, more consistently, upon whether ω_e is defined by equation (3.2) or (3.10). The range of the first subdivision is $\frac{1}{2} < \alpha < 1$; that of the second, $\alpha > 1$. Thus altogether three regions are obtained in the ω-χ plane to which three regions in the ω_e-χ_e plane will correspond. The primitive regions will be referred to as I, II and III, the inverted regions as I-e, II-e and III-e. The numeral defines the correspondence of each. These are as follows:

Primitive region	Range of primitive region in ω-χ plane	Inverted region	Range of inverted region in ω_e-χ_e plane
I	$-\infty < \alpha < \frac{1}{2}$	I-e	$-\infty < \alpha_e < \frac{1}{4}$
II	$\frac{1}{2} < \alpha < 1$	II-e	$0 < \alpha_e < \frac{1}{4}$
III	$1 < \alpha < \infty$	III-e	$-\infty < \alpha_e < 0$

[9] Note again that this notation is deceptively simple since α_e is a function of ω_e.

TABLE 3.1.—FREQUENCY MAPPINGS

Plane		ω-χ	
Primitive region	I	II	III
Range of α	$-\infty < \alpha < \frac{1}{2}$	$\frac{1}{2} < \alpha < 1$	$1 < \alpha < \infty$
Frequency of encounter, ω_e	$\omega_e = \omega(1-\alpha)$	$\omega_e = \omega(1-\alpha)$	$\omega_e = -\omega(1-\alpha)$
Apparent wave heading, χ_e	$\chi_e = \chi$	$\chi_e = \chi$	$\chi_e = \chi + \pi$
Plane		ω_e-χ_e	
Inverted region	I-e	II-e	III-e
Range of α_e	$-\infty < \alpha_e < \frac{1}{4}$	$0 < \alpha_e < \frac{1}{4}$	$-\infty < \alpha_e < 0$
Wave frequency, ω	$\omega = \left[\frac{1-\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e$	$\omega = \left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e$	$\omega = -\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e$
Wave heading	$\chi = \chi_e$	$\chi = \chi_e$	$\chi = \chi_e - \pi$
Jacobian of transformation, $\partial\omega/\partial\omega_e$	$\frac{1}{\sqrt{1-4\alpha_e}}$	$\frac{-1}{\sqrt{1-4\alpha_e}}$	$\frac{1}{\sqrt{1-4\alpha_e}}$

In the above expressions $\alpha = \frac{\omega V \cos \chi}{g}$ and $\alpha_e = \frac{\omega_e V \cos \chi_e}{g}$.

An outline of the frequency mappings for each region is given in Table 3.1.

Since the solution for the wave frequency in the plane is not unique, it is necessary to choose the proper sign before the radical of equation (3.13). The bases for effecting this choice are outlined as follows: Consider first Regions I and I-e. In the primitive region the frequency of encounter is of the same sign as the wave frequency, both being positive. This results from the relation $\omega_e = \omega(1-\alpha)$ and the condition $\alpha < \frac{1}{2}$. This equality of sign must be preserved in the inverted region. The expression in equation (3.13) must consequently result positive for all possible values of $\cos \chi_e$. Since in this case $\cos \chi_e$ can be negative, the expression will remain positive only if the sign before the radical is negative. In Region I-e therefore

$$\omega = \left[\frac{1-\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e \tag{3.18}$$

Similar arguments lead to the relations

$$\omega = \left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e \tag{3.19}$$

in Regions II-e and to

$$\omega = -\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e \tag{3.20}$$

in Region III-e.

THE JACOBIANS

In order to map the product of the response amplitude operator and the spectrum of the seaway into the ω_e-χ_e plane it will be necessary to make use of the Jacobians of the transformations from the ω-χ to the ω_e-χ_e plane. For the pair of equations involved the Jacobian is defined as

$$J\left(\frac{\omega, \chi}{\omega_e, \chi_e}\right) \triangleq \frac{\partial(\omega, \chi)}{\partial(\omega_e, \chi_e)} \triangleq \begin{vmatrix} \frac{\partial\omega}{\partial\omega_e} & \frac{\partial\omega}{\partial\chi_e} \\ \frac{\partial\chi}{\partial\omega_e} & \frac{\partial\chi}{\partial\chi_e} \end{vmatrix} \tag{3.21}$$

and since $\partial\chi/\partial\omega_e = 0$ and $\partial\chi/\partial\chi_e = 1$ the Jacobian in this case reduces to the simple expression

$$J\left(\frac{\omega, \chi}{\omega_e, \chi_e}\right) = \frac{\partial\omega}{\partial\omega_e} \tag{3.22}$$

Expressions for the Jacobians in each region are given in Table 3.1.

The following expressions for the Jacobians are obtained in each region.

Transformation of region	Jacobian
I to I-e	$\frac{1}{\sqrt{1-4\alpha_e}}$
II to II-e	$\frac{-1}{\sqrt{1-4\alpha_e}}$
III to III-e	$\frac{1}{\sqrt{1-4\alpha_e}}$

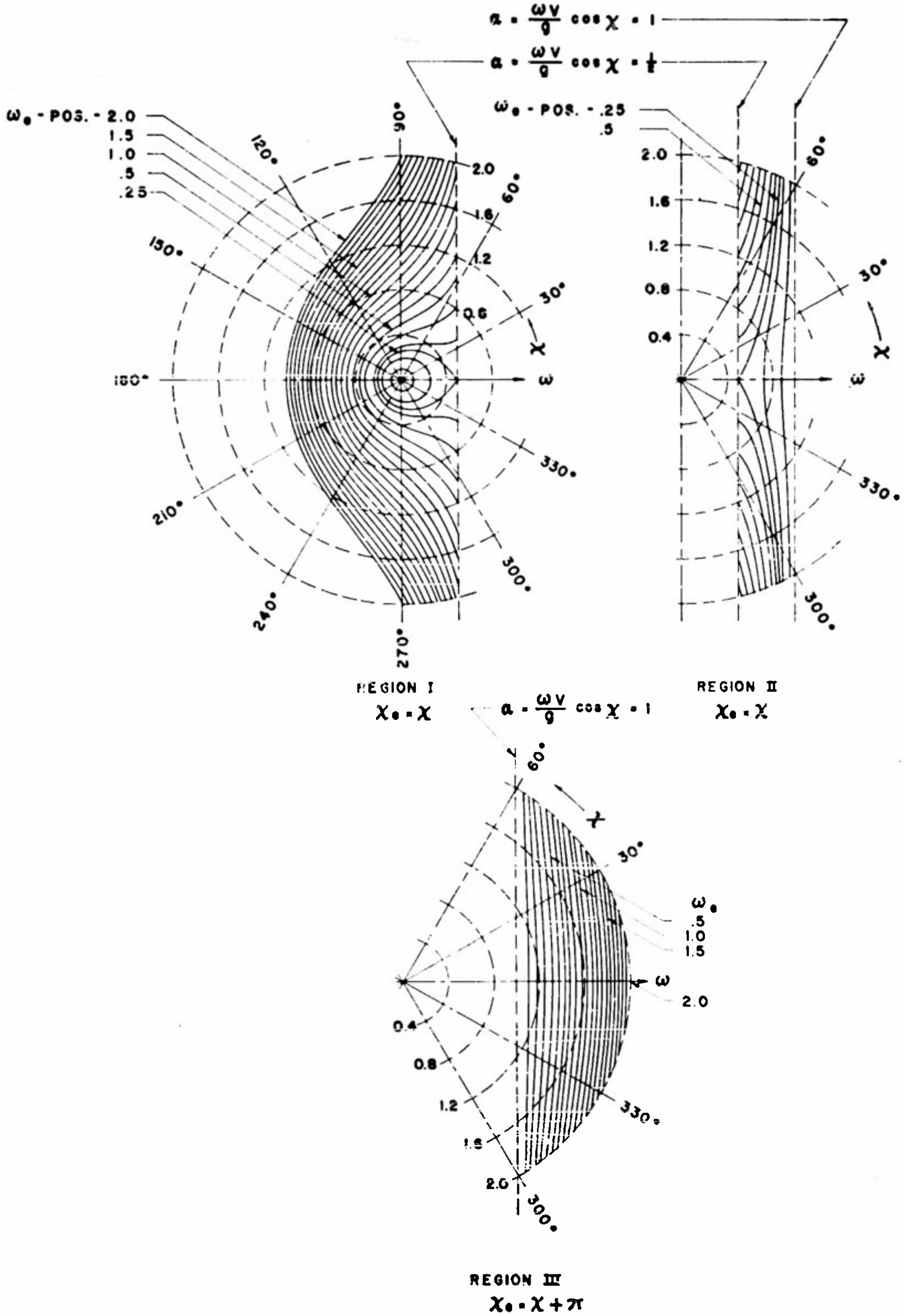


Fig. 3.3.—Regions in the ω-χ Plane Where One-to-One Inverses Exist

See explanatory note for Fig. 3.1.

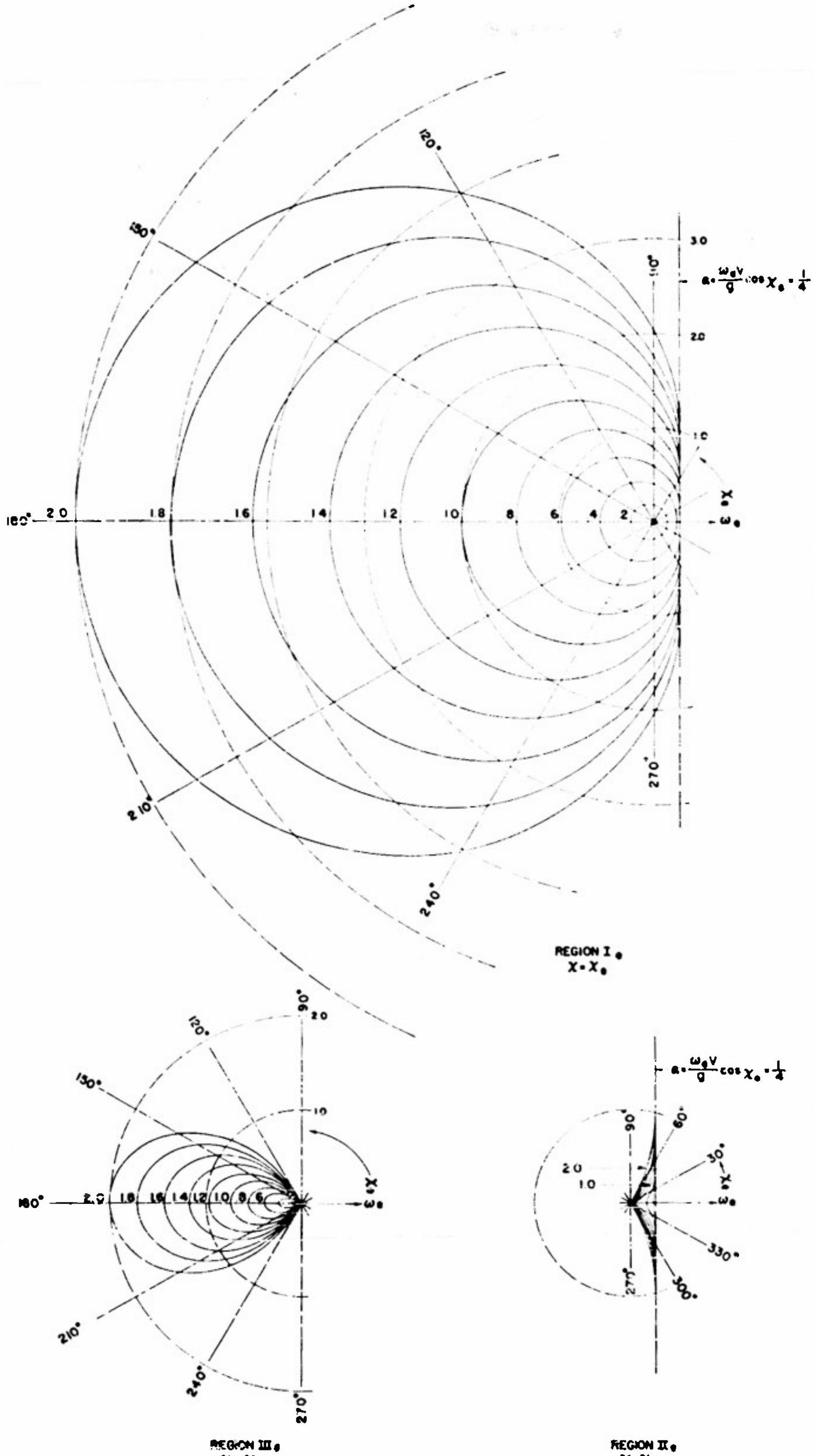
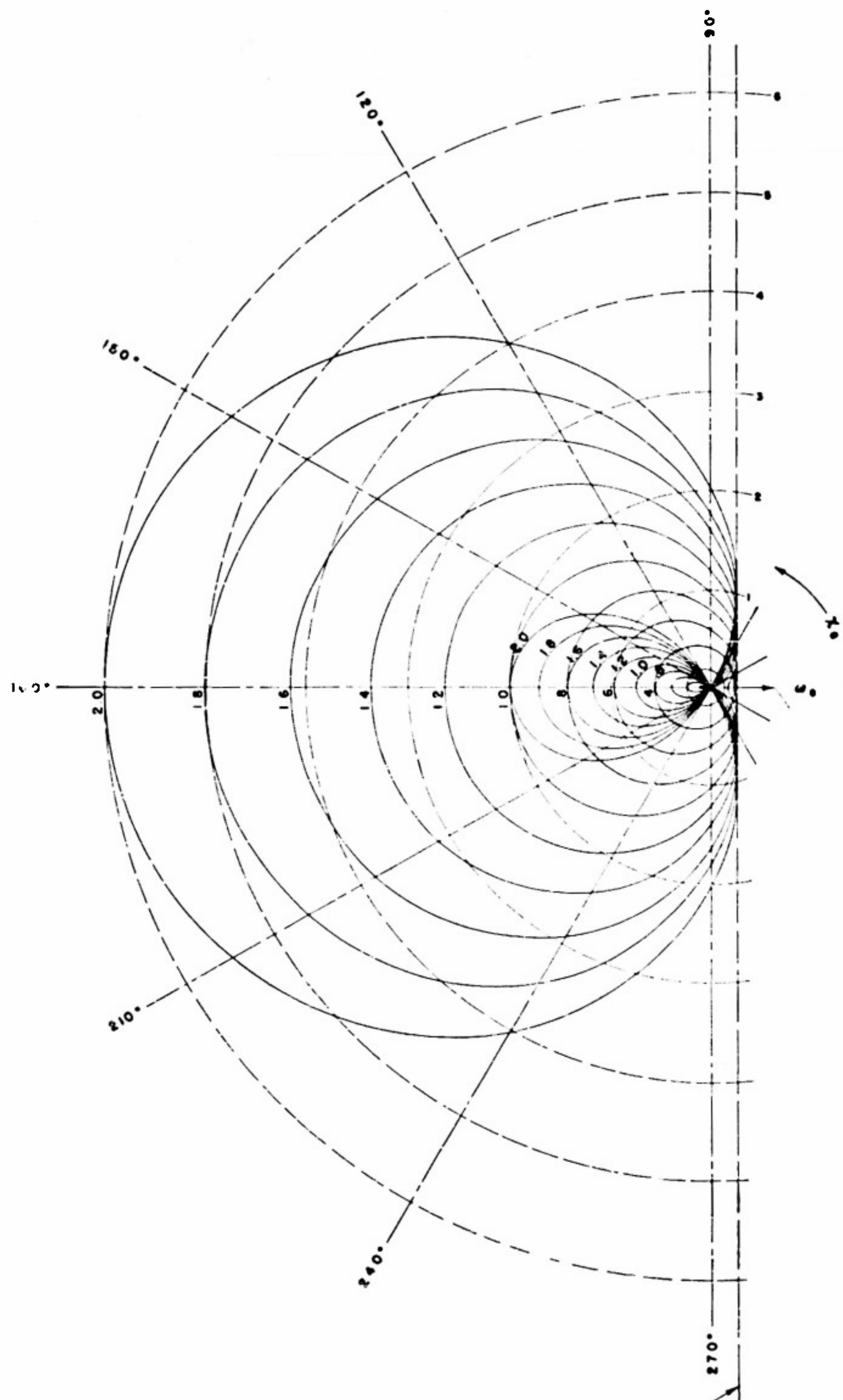


FIG. 3.4.—THE ONE-TO-ONE INVERSES OF THE TRANSFORMATIONS OF THE CURVES IN THE ω-χ PLANE TO THE ω_e-χ_e PLANE

Coordinates and curves are lines of constant $\omega v/g$ and $\omega_e v/g$. Thus to get true values multiply the scale values by g/v. In this figure, circles of constant ω_e have the values, $\omega_e \cdot v/g = 1, 2, 3, 4, 5$ and 6. The curves of constant ω are for constant increments $\omega v/g = 0.20$.



$a = \frac{\omega_e V}{g} \cos \chi_e = \frac{1}{4}$

FIG. 3.5.—LINES OF CONSTANT ω IN THE ω_e-χ_e PLANE

See explanatory note for Fig. 3.4.

SUMMARY

The exposition of this section may be summarized as follows: As a result of the frequency mapping, the original ω-χ plane is divided into three regions. In each region, apart from boundary lines there exists a unique inverse into the ω_e-χ_e plane and its Jacobian can be computed. Over the whole ω-χ plane there are always two related points mapping as a single point in the ω_e-χ_e plane. Or, inversely, to each point in the ω_e-χ_e plane there correspond two points in the ω-χ plane, apart from boundary lines. A complete consistent summary of the frequency mappings is given in Table 3.1. Fig. 3.1 is a plot of constant ω_e curves in the ω-χ plane. In Fig. 3.3 these curves are subdivided into the three regions. In Fig. 3.4 the inverses for each region are shown (curves of constant ω in the ω_e-χ_e plane). In Fig. 3.5 these inverses are all combined.

THE SHIP RESPONSES

In the second section of this paper a statistical description of the seaway was given in terms of energy spectra. The third section was a discussion of the response amplitude operators for heave, pitch and roll in a regular seaway. These were each obtained as products of a force and a magnification factor. Both of these factors were expressed as functions of certain ship parameters and of the wave frequency, ω, and wave heading, χ, referred to a fixed system of coordinates (system c). The fourth section was devoted to mapping the frequencies of the ship's response when in a regular seaway from the fixed coordinate system to the coordinate system moving with the vessel. In the present section all the foregoing development will be combined to give the response of a vessel in heave, pitch and roll to the statistically described irregular seaway of the second section. Exposition of the procedure will be carried out for the generalized motion s which may be interpreted as being either heave, pitch or roll, the steps being identical for all motions. One will begin by considering the response to a regular sea composed of a simple sine wave.

REGULAR SEAWAY

Consider a ship progressing at a velocity v in a simple sinusoidal seaway which is characterized by an amplitude, r_1, a wave frequency, ω_1, a direction relative to the ship's course, χ_1, and a phase lag ϵ_1. The square of the amplitude of the response in any motion s is given by

$$[s_1]^2 = [r_1]^2 \cdot [A_s(\omega_1, \chi_1, v)]^2 \qquad (4.1)$$

and the response lags the excitation by a phase shift, ϵ_s.

For a given value of ω_1, χ_1 and v there is one, and only one, value of ω_e given by equation (3.2). Let this value be $(\omega_e)_1$. Then the response of the ship to the simple sinusoidal wave system is given by

$$s(t) = \sqrt{[r_1]^2 \cdot [A_s(\omega_1, \chi_1, v)]^2} \cdot \cos((\omega_e)_1 t + \epsilon_1 + \epsilon_s) \qquad (4.2)$$

CONFUSED SEAWAY

Consider now a more complex seaway formed by the sum of a number of very low simple harmonic progressive waves. This seaway is represented by the partial sum of equation (1.19) which in the limit gives the short-crested Gaussian sea surface. Each term of the sum is treated like the single sine wave discussed above. Since the theory is linear, *it is assumed that the sum of the responses of a ship to a number of simple sine waves is equal to the response of the ship to the sum of the waves.* This is the fundamental assumption of the procedure expounded herein. If it is not valid, then the results derived in this paper are not valid. Although this assumption appears justifiable at least for average motions which are not too great, it has not been possible to develop a rigorous proof of its validity within the linear theory.

If in the partial sum defining the seaway the squares of the wave amplitudes are multiplied by the amplitude response operators, the following equation results for the sum of the squares of the amplitude of the responses.

$$[s(\omega, \chi, v)]^2 = \sum_{m=0}^{r} \sum_{n=0}^{q} [r(\omega_{2n+1}, \chi_{2m+1})]^2 \cdot [A_s(\omega_{2n+1}, \chi_{2m+1}, v)]^2 \cdot [(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})] \qquad (4.3)$$

The limit of this sum is defined by a symbol R_s^* which is similar in concept to the symbol R defined by equation (1.12). Thus

$$R_s^* \triangleq \int_0^\infty \int_{-\pi}^{\pi} [r(\omega, \chi)]^2 \cdot [A_s(\omega, \chi, v)]^2 \, d\chi \, d\omega \tag{4.4}$$

A series of readings from the record of the ship's response will have a normal distribution. This will be the same as given by equation (1.16) with R_s^* substituted for R.[10]

This step may be extended to obtain the ship's response as a function of time to the complex seaway defined by equation (1.19). The expression, which will parallel equation (4.2), is obtained as follows:

For each term of the sum in equation (4.3) there is a unique value of ω_e. Let this value be designated as $\omega_e(\omega_{2n+1}, \chi_{2m+1})$.

Due to the random phases occurring in the partial sum for the seaway, equation (1.19), it is convenient to lump all the terms which are invariant with respect to time into a new phase lag. This may be designated $\epsilon_s'(\omega_{2n+1}, \chi_{2m+1})$ and its distribution will still be in accordance with equation (1.13). The response of the ship to the multiple and randomly superposed sinusoidal wave system is then

$$s(t) = \sum_{m=0}^{r} \sum_{n=0}^{q} \left\{ [r(\omega_{2n+1}, \chi_{2m+1})]^2 \cdot [A_s(\omega_{2n+1}, \chi_{2m+1}, v)]^2 \cdot [(\omega_{2n+2} - \omega_{2n})(\chi_{2m+2} - \chi_{2m})] \right\}^{1/2} \cdot \cos [\omega_e(\omega_{2n+1}, \chi_{2m+1})t + \epsilon_s'(\omega_{2n+1}, \chi_{2m+1})] \tag{4.5}$$

This is the final step in the derivation. For a small enough net in the ω—χ plane, equation (4.5) yields a ship response which is a function of time and which is Gaussian. For this net all of the terms in the partial sum which are associated with a frequency of encounter within a finite but small range can be treated as if they gave rise to a single sinusoidal term of a frequency corresponding to that obtaining at the center of the range. By arguments similar to those following equation (1.23), the response spectrum of the ship is then known as a function of ω_e. This step can be carried out more simply for the continuous spectrum as outlined below.

The Spectrum of the Response

This representation, given by equation (4.5), however, is as yet unsatisfactory. The response of a ship in one of her degrees of freedom is purely a function of time and should be represented as such. The response should, consequently, have an integral representation like that of a wave record when obtained as a function of time at a fixed point, equation (1.10). To this end the energy spectrum of the response will be derived. The step-by-step procedure for obtaining the spectrum is as follows:

(1) The energy spectrum of the seaway referred to the ω-χ plane is divided into the three parts corresponding to Regions I, II and III. This step is to the end of obtaining unique inverses when mapping into the ω_e-χ_e plane. The following terms consequently are obtained.

$$[r(\omega, \chi)]^2 \triangleq [r_I(\omega, \chi)]^2 + [r_{II}(\omega, \chi)]^2 + [r_{III}(\omega, \chi)]^2 \tag{4.6}$$

where each spectrum is identically zero outside the region of its definition.

(2) Each term of the energy spectrum of the seaway is multiplied by the square of the response amplitude operator. The result of this step is related to the square of the amplitude of the response by virtue of equation (4.4). This gives

$$[r_I(\omega, \chi)]^2 [A_s(\omega, \chi, v)]^2 + [r_{II}(\omega, \chi)]^2 [A_s(\omega, \chi, v)]^2 + [r_{III}(\omega, \chi)]^2 [A_s(\omega, \chi, v)]^2 \tag{4.7}$$

(3) Each primitive region in the ω-χ plane is now mapped into the corresponding region in the ω_e-χ_e plane. This mapping is carried out separately for each region as follows:

Region	For ω write	For χ write
I to I-e	$\left[\dfrac{1 - \sqrt{1 - 4\alpha_e}}{2\alpha_e}\right]\omega_e$	χ_e
II to II-e	$\left[\dfrac{1 + \sqrt{1 - 4\alpha_e}}{2\alpha_e}\right]\omega_e$	χ_e
III to III-e	$\left[\dfrac{-1 - \sqrt{1 - 4\alpha_e}}{2\alpha_e}\right]\omega_e$	$\chi_e - \pi$

[10] This statement has been verified on a number of different ship motion records.

The outcome is

$$\left[r_{\mathrm{I}}\left(\left[\frac{1-\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e\right)\right]^2\cdot\left[A_s\left(\left[\frac{1-\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e\right)\right]^2$$
$$+\left[r_{\mathrm{II}}\left(\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e\right)\right]^2\cdot\left[A_s\left(\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e\right)\right]^2$$
$$+\left[r_{\mathrm{III}}\left(-\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e-\pi\right)\right]^2\cdot\left[A_s\left(-\left[\frac{1+\sqrt{1-4\alpha_e}}{2\alpha_e}\right]\omega_e,\ \chi_e-\pi\right)\right]^2 \quad (4.8)$$

The foregoing expression is a function of ω_e and χ_e such that the value of equation (4.7) assigned to the original point, $\omega_{\mathrm{I}}, \chi_{\mathrm{I}}$, is now transferred to a new point, $\omega_{e\mathrm{I}}, \chi_{e\mathrm{I}}$. The notation can be simplified by the use of the fact that as far as the mapping is concerned, equation (4.8) is just a function of two variables and the result is

$$[r_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,[A_{se}(\omega_e,\chi_e)]^2 + [r_{\mathrm{II}\text{-}e}(\omega_e,\chi_e)]^2\,[A_{se}(\omega_e,\chi_e)]^2 + [r_{\mathrm{III}\text{-}e}(\omega_e,\chi_e)]^2\,[A_{se}(\omega_e,\chi_e)]^2 \quad (4.9)$$

where the functions are defined in the appropriate regions of the ω_e-χ_e plane as determined from Table 3.1.

(4) The integral over ω_e and χ_e of equation (4.9) is, however, no longer equal to R_s^*. To make this integral attain the value R_s^* each term must be multiplied by the appropriate Jacobian. The result is

$$\frac{(r_{\mathrm{I}\text{-}e}(\omega_e,\chi_e))^2\,[A_{se}(\omega_e,\chi_e)]^2}{\sqrt{1-4\alpha_e}} - \frac{[r_{\mathrm{II}\text{-}e}(\omega_e,\chi_e)]^2\,[A_{se}(\omega_e,\chi_e)]^2}{\sqrt{1-4\alpha_e}} + \frac{[r_{\mathrm{III}\text{-}e}(\omega_e,\chi_e)]^2\,[A_{se}(\omega_e,\chi_e)]^2}{\sqrt{1-4\alpha_e}} = [s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2 - [s_{\mathrm{II}\text{-}e}(\omega_e,\chi_e)]^2 + [s_{\mathrm{III}\text{-}e}(\omega_e,\chi_e)]^2 \quad (4.10)$$

where the definition of the new symbols is evident from the notation.

(5) The spectrum of the ship's response is obtained by integration over χ_e of equation (4.10). The sense of integration must be carefully considered. It must be always counterclockwise over the angular variables and always toward increasing frequencies.

In Region I integration over ω ranges from zero to an upper limit defined by $\alpha = \frac{1}{2}$ (for cos χ positive) or by infinity (for cos χ negative). The corresponding limits in Region I-e are from zero to $\alpha_e = \frac{1}{4}$ or to infinity, respectively. Therefore the integration over both ω and ω_e is in the same sense in both planes.

In Region II the lower and upper limits of ω are given by $\alpha = \frac{1}{2}$ and $\alpha = 1$ for a fixed χ. These correspond, respectively, to lower and upper limits for ω_e in II-e given by $\alpha_e = \frac{1}{4}$ and $\alpha_e = 0$ for a fixed χ_e. The sense of integration over ω_e in II-e is now opposite that in II. The direction of integration over ω_e can, however, be reversed simply by changing the sign of the integrand, i.e., by changing into positive the negative sign before the second term of equation (4.10).

In Region III the lower limit of ω is given by $\alpha = 1$ for a fixed χ and the upper limit is infinity. These limits correspond to lower and upper limits in III-e given by $\omega_e = 0$ and $\omega_e = +\infty$. Therefore the integration is in the same sense in both regions.

The spectrum of the ship response is thus defined by

$$[s(\omega_e)]^2 = \int_{\mathrm{I}\text{-}e}[s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e + \int_{\mathrm{II}\text{-}e}[s_{\mathrm{II}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e + \int_{\mathrm{III}\text{-}e}[s_{\mathrm{III}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e \quad (4.11)$$

In this equation the direction of integration is always counterclockwise. The limits of integration may be functions of ω_e. This will occur when $\omega_e > g/4v$. For example, the integral over Region I-e is expressed more completely as

$$\int_{-\pi}^{0}[s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e + \int_{0}^{\pi}[s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e$$

for $\omega_e < g/4v$

$$+\int_{-\pi}^{\arccos(g/4v\omega_e)}[s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e + \int_{\arccos(g/4v\omega_e)}^{\pi}[s_{\mathrm{I}\text{-}e}(\omega_e,\chi_e)]^2\,d\chi_e \quad (4.12)$$

for $\omega_e > g/4v$

This equation shows that Region I-e is integrated over the full circular range of χ_e for values of ω_e which lie within a circle of radius $g/4v$. It is integrated over a partial circle of χ_e values otherwise, since the line, $\alpha_e = \frac{1}{4}$, is a boundary of the region, and the limits of integration are then variable as a function of ω_e.

(6) Equation (4.11) now permits the definition of the cumulative density as

$$R_s^*(\omega_e) \triangleq \int_0^{\omega_e}[s(\omega_e)]^2\,d\omega_e \quad (4.13)$$

and of R_s^* as

$$R_s^* \doteq \int_0^\infty [s(\omega_e)]^2 \, d\omega_e \qquad (4.14)$$

This is the same number as given by equation (4.4).

Thus it is seen that the product of the energy spectrum of the seaway and an amplitude response operator gives the correct root-mean-square value of the motion on the basis that its statistical parameters are determined in terms of the normal distribution. Such an operation, however, does not provide the correct shape of the energy spectrum of the response as a function of ω_e. The correct distribution is obtained through a frequency mapping which preserves the value of the cumulative energy density of the response R_s^*. For additional details on the theory of mappings the reader is invited to read the text by Courant, Vol. II, Chapter III, Sections 2, 3 and 4 [4].

A remark may be made on the dimensions of the energy spectrum for the motions considered. The energy spectrum in heave has the dimensions of length squared time whereas the energy spectra in pitch and roll have the dimensions time.

THE SHIP MOTIONS

"... to pursue mathematical analysis while at the same time turning one's back on its applications and on intuition is to condemn it to hopeless atrophy." —R. COURANT

As a result of the development in the previous sections the equations for the heaving, pitching and rolling motions of a ship can be written as

$$\begin{aligned} z(t) &= \int_0^\infty \cos\,[\omega_e t + \epsilon(\omega_e)]\,\sqrt{[z(\omega_e)]^2\,d\omega_e} \\ \psi(t) &= \int_0^\infty \sin\,[\omega_e t + \epsilon(\omega_e)]\,\sqrt{[\psi(\omega_e)]^2\,d\omega_e} \\ \varphi(t) &= \int_0^\infty \sin\,[\omega_e t + \epsilon(\omega_e)]\,\sqrt{[\varphi(\omega_e)]^2\,d\omega_e} \qquad (5.1) \end{aligned}$$

The definition of the integrals in these equations is the same as that of the integral of equation (1.14). It consequently follows that the ship motions have the same properties as the wave records. This conclusion opens up for application to the study of ship motions mathematical developments obtained not only in the field of wave theory, as, e.g., the work of Longuet-Higgins, to mention but one researcher, but also in allied fields. The procedures developed in electronic theory by Wiener, Rice and Tukey and in statistical theory by Doob, Lévy and Cramér, to name only a few, may be cited as examples. Some of the conclusions and properties applicable to ship motions and derived from these works are briefly stated below. A study of these references will undoubtedly yield additional aspects.

NUMERICAL DETERMINATION OF THE SPECTRUM

In developing the theory of this paper, the energy spectrum of the seaway was taken as the point of departure and the energy spectrum of the response of a ship was derived through application of ship response amplitude operators and a frequency transformation. This is not the only way to obtain the response energy spectrum. If the motions of the ship are recorded, their energy spectrum can be obtained directly. The procedure rests upon the knowledge of the auto-correlation-function for the motions involved.

The non-normalized auto-correlation function is defined as[11]

$$Q_s(h) \triangleq \lim_{t_n \to \infty} \frac{2}{t_n} \int_{t_0}^{t_0+t_n} s(t)s(t+h)\,dt \qquad (5.2)$$

where t_n is the duration of the record analyzed, t_0 is the initial time of the record, h is an increment of time. The energy spectrum of a ship's response is then given by the Fourier cosine transform of the non-normalized auto-correlation function of the response. Symbolically,

$$[s(\omega_e)]^2 = \frac{2}{\pi} \int_0^\infty Q_s(h) \cos w_e h \; dh \qquad (5.3)$$

These expressions imply two conditions:

(a) That the available record of the motion is infinitely long;

(b) That the response is "stationary" in the sense that its fundamental character (as a multivariate Gaussian distribution) does not change.

These conditions are, of course, not met in nature. However, for the purposes of deducing the energy spectrum, it is sufficient that the record remain homogeneous only for a reasonable length of time. Since waves from a storm at sea change their character slowly over intervals of the order of hours, this practical requirement is generally fulfilled.

The procedure for obtaining the energy spec-

[11] The true auto-correlation function with unity as the value at lag zero is found by dividing equation (5.2) by R_s^*, since $Q_s(0)$ as defined above can be shown to be equal to R_s^*.

trum from an actual record of finite duration has been given by Tukey and by Tukey and Hamming [22] [23]. Their method consists in replacing the integral of equation (5.3) by a partial sum, takes into account the errors entering into a numerical analysis of a record of finite duration and applies corrections to the "raw" results. The procedure applied to a ship's response is briefly summarized as follows:

(a) The finite record of the response is subdivided into equally spaced intervals, the increment being Δt seconds where Δt should be approximately one half of the smallest relevant period in the spectrum. Then

$$t_1 - t_0 = t_2 - t_1 = \ldots t_n - t_{n-1} \triangleq \Delta t \quad (5.4)$$

(b) At each subdivision the value of response in terms of departure from the mean is tabulated forming the sequence

$$s(t_1),\ s(t_2)\ \ldots\ s(t_n) \quad (5.5)$$

(c) The non-normalized auto-correlation coefficients are found by

$$Q_s(h) = \frac{2}{n-h} \sum_{q=1}^{n-h} s(t_q)\, s(t_{q+h}); \qquad (h = 0, 1, 2 \ldots m) \quad (5.6)$$

where m represents a lag. There are $m + 1$ such coefficients as $Q_s(h)$ ranges from $Q_s(0)$ to $Q_s(m)$.

(d) The "raw" estimates of energy in the band

$$\frac{\pi(k - \frac{1}{2})}{\Delta t \cdot m} < \omega_e < \frac{\pi(k + \frac{1}{2})}{\Delta t \cdot m}$$

are found by

$$L_k = \frac{1}{m}\left[Q_s(0) + 2\sum_{h=1}^{m-1} Q_s(h) \cos\frac{\pi hk}{m} + Q_s(m) \cos \pi k\right]; \qquad (k = 0, 1, 2 \ldots m) \quad (5.7)$$

There are again $m + 1$ such values.

(e) The "raw" estimates are now corrected for the effect of distortions introduced by having based the computation on values taken at discrete points of a finite record. The correction process gives corrected energy coefficients

$$U_k = 0.23\, L_{k-1} + 0.54 L_k + 0.23 L_{k+1} \quad (5.8)$$

where $L_{-1} = L_1$ and $L_{m+1} = L_{m-1}$

(f) At the frequency $\omega_e = (\pi k/\Delta t \cdot m)$ the best estimate of the value of $[s(\omega_e)]^2$ is given by

$$[s(\omega_e)]^2 = \frac{U_k \Delta t \cdot m}{\pi} \quad (5.9)$$

The reliability of the results obtained by this method depends on the degrees of freedom of the system defined by

$$f \triangleq \frac{n - m/4}{m/2} \quad (5.10)$$

The accuracy increases with the degrees of freedom, as is obvious, for a high degree of freedom implies a very long record (n) and a not too high resolution (m). The accuracy is determined from the chi-square distribution with f degrees of freedom from which the 90% confidence limits of $[s(\omega_e)]^2$ can be found.

Short Range Prediction

The auto correlation function can be used to predict the response of the ship in the immediate future. The prediction is a least square prediction in that the square of the difference between the predicted and the observed motion is a minimum. The accuracy of the forecast for a given time h in the future depends on the normalized values of $Q_s(h)$ beyond h. The lower this value, the lesser the accuracy. A corollary of this is that a motion record with a narrow band energy spectrum can be forecasted into the future more accurately than a motion having a wide band energy spectrum.

For typical motion records the auto-correlation function becomes essentially zero after about 100 seconds. This means that no short range prediction is possible beyond this interval of time. The statistical properties are still in a sense predictable, although the shape of the future record is unknown.

Envelope, Maxima, and Zeroes

The properties of wave records deduced from the study by Rice apply in their entirety to ship motions. Thus formulae for the average amplitudes of a motion $s(t)$ corresponding to equations (1.31), (1.32) and (1.33) can be written down immediately giving

(a) The average amplitude of motion

$$\bar{s} = 0.866\sqrt{R_s^*} \quad (5.11)$$

(b) The significant amplitude of motion

$$\bar{s}_{1/3} = 1.415\sqrt{R_s^*} \quad (5.12)$$

(c) The average of the $^1/_{10}$ highest amplitudes

$$\bar{s}_{1/10} = 1.80\sqrt{R_s^*} \quad (5.13)$$

For example, if $R_\varphi^* = 0.01$ radian squared the mean roll amplitude would be 0.0866 radian or 4.96° and the average of the $^1/_{10}$ highest rolls would be 10.3°.

Since our application of this particular aspect of the work of Rice is based on the assumption of a narrow spectrum, the foregoing formulae apply with somewhat greater validity when the vessel finds herself in sea conditions approaching a swell.

The probability curve for roll amplitudes given by Perry in the discussion to the paper by Weinblum and St. Denis is essentially given (apart from the abscissa) by $e^{-\varphi^2/R_\varphi^*}$ which is one minus the cumulative distribution function whose probability distribution is given by equation (1.30) with $\bar{R}_\varphi^*$ substituted for R. In this expression φ is the amplitude of roll and the number that results is the probability of a roll in excess of φ.

When in a regular seaway a vessel undergoes an oscillatory rolling motion about the mean value zero such that the number of times she attains a maximum amplitude to the one side or other equals exactly the number of zero crossings. In a confused sea such is no longer the case. The motion being erratic, the ship can, for example, roll to a maximum amplitude, start back to the angle of zero roll, but, before attaining it, reverse the direction and roll out again. The number of roll maxima to both sides is consequently greater than the number of zero crossings. The study of Rice permits one to determine both the number of zero crossings per second as well as the number of times per second the maximum amplitude is attained to either side.

The number of zero crossings per second for the motion s is given by

$$2\bar{f}_s = \frac{1}{\pi}\left[\frac{\int_0^\infty \omega_e^2[s(\omega_e)]^2\,d\omega_e}{\int_0^\infty [s(\omega_e)]^2\,d\omega_e}\right]^{1/2} \qquad (5.14)$$

where the reciprocal of f_s is the average period of the motion s in the loose sense of the word "period."

As against this the number of times per second a maximum amplitude to both sides is attained is given by

$$2(\bar{f}_s)_m = \frac{1}{\pi}\left[\frac{\int_0^\infty \omega_e^4[s(\omega_e)]^2\,d\omega_e}{\int_0^\infty \omega_e^2[s(\omega_e)]^2\,d\omega_e}\right]^{1/2} \qquad (5.15)$$

For heave and pitch equation (5.15) gives the total number of maxima and minima per second.

In spite of the identity of the frequency of encounter for all simultaneous motions of a vessel, the number of zero crossings per second and maximum amplitudes per second may vary appreciably with the motion. This is illustrated by Fig. 5.1 which is a simultaneous record of roll and pitch for a ship underway in a seaway. The average frequency of roll is much smaller than that of pitch. This would be inconsistent if the vessel were in a regular seaway. However, this phenomenon can be explained logically on the basis of the theory presented herein.

The spectrum of the seaway is the same for both motions. But the response amplitude operators for roll and pitch are quite different in form. When these are applied to the seaway a different spectrum of response amplitude is obtained for each motion in both the ω-χ and ω_e-χ_e planes. As a consequence the values of $\bar{f}_\varphi$ and $\bar{f}_\psi$ obtained from equation (5.14) are entirely different.

Extreme Motions

At times it is desirable to estimate the average value of the highest expected amplitude of motion, say roll, out of a total of N recorded oscillations.

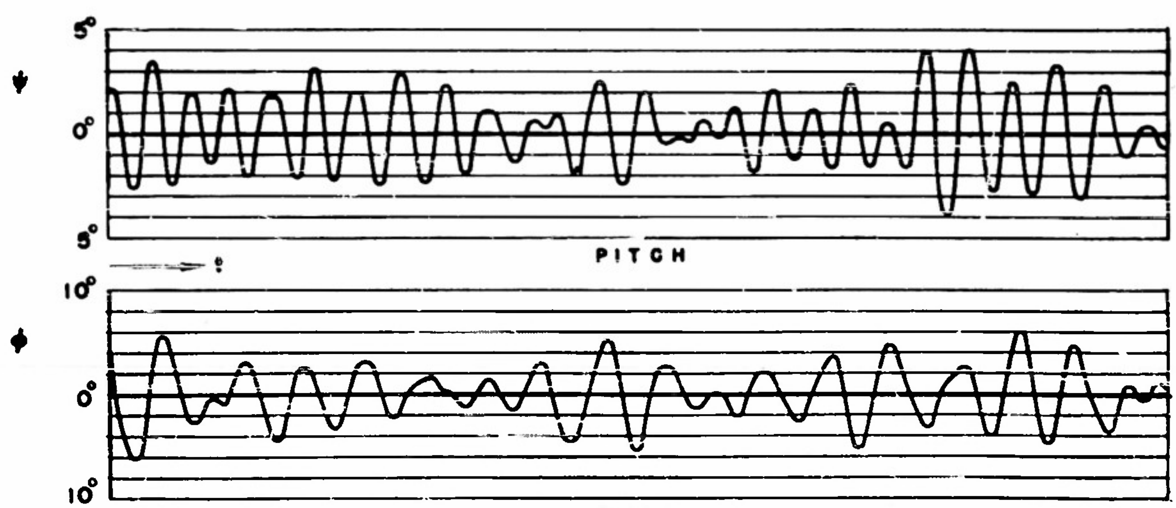


Fig. 5.1.—Simultaneous Record of Roll and Pitch for a Ship Underway in a Seaway

This is possible through a technique developed by Longuet-Higgins for the analysis of waves [10]. The method rests on the assumption that the envelope of a wave or motion record passes through all the peaks of the record itself. This implies that the spectrum is narrow. Nevertheless the results are roughly applicable to confused sea conditions.

Longuet-Higgins derived the probability distribution function for the amplitude of the highest wave in a wave record N waves long. This derivation is based on obtaining the probability distribution function for the envelope of the wave record, see Fig. 1.9. If the latter is known then the expected value for the amplitude of the highest wave out of a total of N waves can be derived. This, of course, applies directly to ship motions. A tabulation may be thus made of the expected value of the highest amplitude of motion out of a total of N oscillations.

Number of oscillations, N	Amplitude of motion
20	$1.87 \times \sqrt{R_r^*}$
50	$2.12 \times \sqrt{R_r^*}$
100	$2.28 \times \sqrt{R_r^*}$
200	$2.43 \times \sqrt{R_r^*}$
500	$2.60 \times \sqrt{R_r^*}$
1000	$2.73 \times \sqrt{R_r^*}$

Consider roll for example. If $R_r^* = 0.01$ radian squared then, out of 50 rolls, one roll should have an amplitude of 12.2 degrees. Out of a total of 500 rolls, one roll should attain an amplitude of 14.8 degrees. The value of the highest amplitude increases like $[\log N]^{1/2}$.* Even for very high values of N the results appear to be reliable. However, as often occurs in statistics, the method eventually breaks down by predicting too extreme values. This occurs because in practical application the theoretical distribution cannot be relied upon at the extreme ranges of the function.

Conclusions

In this paper the authors have attempted to show that theoretical studies of ship motions need not be confined to those experienced by a vessel in a seaway whose pattern is rhythmic and regular and, therefore, unreal. The motions that a vessel undergoes in a confused seaway—such as occurs within, or close to, a storm generating area—can be derived provided one seeks only knowledge as to the amplitudes of motion and foregoes (for the time being) any pursuit of phase relationships between vessel and sea. It is possible to treat phase relationships by more advanced techniques, but then the mathematics become extremely difficult.[12] In confused seas phase relations are of lesser importance in most practical applications. Confining the interest to amplitudes alone makes for a powerful extension of the available theory of ship motions. It then becomes possible to make direct use of a statistical definition of the seaway based on its energy spectrum. Through application of the proper response amplitude operators and frequency mappings the statistical definition of the seaway is converted into a quasi-homogeneous Gaussian process with a known spectrum. An immediate consequence is a statistical definition of the motions of the vessel for any course and speed. Indeed the root-mean-square values of the motions and a measure of the occurrence of extreme amplitudes are obtained in this manner. However, these are quite sufficient to define the real motions that a ship experiences in a real sea.

Based on the fact that the process is very nearly Gaussian, the behavior of a ship is characterized by equation (5.1). This permits the direct application of all that is known about Gaussian processes to the study of ship motions. Many properties of ship motions, such as obtained in this section with relative ease, are immediately derivable from the assumption of a Gaussian motion and from the knowledge of the spectrum.

The essential idea expounded in this paper is that the motions of a ship are a Gaussian process and that they are completely characterized by a response spectrum defined by equation (4.11). This representation leads to the conception that the response of a ship in a confused sea is a steady-state process rather than a continuous succession of transients. The advantages to be gained from such a conception are powerful for they lead immediately to practical results.

The findings of this paper may be summarized as follows: The representation of the oscillatory motions of a ship in confused seas by a response spectrum based on a statistical definition of the seaway is sufficient to give a characterization of these motions of such completeness that the naval architect may obtain practical solutions to many problems in this field which he now deems important.

[12] To treat phase relationships it would be necessary to know $\epsilon_s(\omega, \chi)$ in addition to $A_s(\omega, \chi)$ in analytic detail. Then the ship motion could be correlated with a set of neighboring points in the seaway after it had been mapped into the $\omega_e - \chi_e$ plane. Although the phase of each component wave is random, the phase lag between each component wave and the ship response is a definite function. The correlation between a passing wave and an individual cycle of the motion could then be found.

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NOTATION

SYMBOLS

a	coefficient
b	coefficient
	variable of integration
c	wave velocity
$c(X)$, $c(\xi)$	linearizing coefficient
c_f	frictional coefficient
c_s	separation coefficient
e	exponential
f	degrees of freedom
f_s	frequency for motion
f_s, f_a	excitation coefficients (symmetric and antisymmetric)
g	acceleration of gravity
h	height of waterline above origin
	wave height
	increment of time
h_s	damping coefficient for motion
k	wave number
	boundary of interval
k_x, k_y, k_z	linear inertia coefficients
k_{xx}, k_{yy}, k_{zz}	angular inertia coefficients
l	girth
m, n	transverse and vertical direction cosines
p	probability
q	strength of source
r	wave ordinate
s	displacement
	general motion
	(x = surge, y = sway, z = heave, φ = roll, ψ = pitch, θ = yaw)
$s(X)$, $s(\xi)$	linearizing coefficient
$s^2(X)$, $s^2(\xi)$	linearizing coefficient
t	time
v	ship's speed
w	width of bilge keels
x, y, z	motions along principal axes surge, sway and heave, respectively
A_s	response amplitude operator for motion s
$A(X)$	cross-sectional area
$A(Z)$	waterplane area
B	ship's beam
C_s	hydrodynamic inertia coefficient in motion s
$C_s(X)$, $C_s(\xi)$	sectional value of C_s
$(C_l)_s$	aspect ratio correction in motion s
E	energy
E_s	force factor for motion s
F	force
	amplitude of heaving excitation
G_z^s, G_z^a, G_ψ^s, G_ψ^a and G_φ^s, G_φ^a	symmetric and antisymmetric excitation functions in heave, pitch and roll, respectively
$\overline{GM}$	metacentric height
H	ship's draft
I_x, I_y, I_z	moment of inertia of ship about principal axes
I_{xx}, I_{yy}, I_{zz}	hydrodynamic moment of inertia
I_ψ, I_φ	virtual moment of inertia in pitch and roll
J_x, J_y, J_z	moment of inertia of volume of displacement about principal axes
L	ship's length
	"raw" energy coefficient
M	moment
M_z	virtual mass in heave
M_φ, M_ψ	virtual moment of inertia in roll and pitch
N	integer
N_s	coefficient of damping in motion s
$N_s(X)$, $N_s(\xi)$	sectional value of damping coefficient
O	origin of coordinates
$Q_s(h)$	auto-correlation function for motion s
R	cumulative energy density
R_s	restoration coefficient for motion s (appendix)
R_s^*	cumulative response amplitude density in motions
U	corrected energy coefficients
WL	waterline
X, Y, Z	Cartesian coordinates
$\bar{Y}$, $\bar{Z}$	horizontal and vertical levers to centroid of immersed volume
α	waterplane coefficient (appendix)
	ratio of effective ship's speed to wave velocity (text)
$\beta(X)$, $\beta(\xi)$	sectional coefficient
γ	wave-length parameter
δ	phase lag
ϵ	phase lag of excitation
ϵ_s	phase lag of ship's response in motion s
ζ, η, ξ	dimensionless Cartesian coordinates $\zeta \triangleq -Z/H$, $\eta \triangleq 2Y/B$, $\xi \triangleq 2X/L$

θ . . . . angular coordinate
κ_s . . . . dimensionless damping coefficient in motion s
λ . . . . wave length
μ_s . . . . magnification factor in motion s
ν_s . . . . natural undamped frequency of oscillation of vessel in motion s
π 3.1416
ρ . . . . density of water
σ . . . . vertical coefficient
τ . . . . wave period
φ . . . . roll angle
χ . . . . direction of wave travel
ψ . . . . pitch angle
ω . . . . wave frequency
ω_e . . . . frequency of wave encounter
Λ . . . . tuning ratio
∇ . . . . volume of displacement

SUBSCRIPTS

a . . . . . . absolute
. . . antisymmetric
b . . . . . . referring to bilge keels
e . . . . . . effective or encounter
f . . . . . . frictional
j . . . . . . integer
k . . . . . integer
m . . . . . maximum
. . . . . integer
n . . . . . integer
o . . . . . . referring to the origin
. . . . . referring to the surface
p . . . . . integer
q . . . . . . integer
r . . . . . . relative
s . . . . . symmetric
. . . . . . separation
. . . . . referring to the hull's surface
. . . . . referring to motion s
. . . . . . integer
v . . . . . . viscous
w . . . . . . wind
. . . . wave
z . . . . . . referring to heave
ψ . . . . . referring to pitch
φ . . . . . referring to roll

APPENDIX

DERIVATION OF THE RESPONSE AMPLITUDE OPERATORS

The response amplitude operators are obtained as solutions to the equations of motion. The latter are obtained for a particular motion by equating the sum of the inertial, the damping and the restoring reactions to the excitation. In their linearized form and on the basis of the assumptions made in the section on the response amplitude operators, page 297, the equations of motion for heave, pitch and roll may be respectively written as follows:

$$\left.\begin{aligned} M_z \frac{d^2z}{dt^2} + N_z \frac{dz}{dt} + R_z z &= F_z \cos(\omega_e t - \epsilon_z) \\ I_\psi \frac{d^2\psi}{dt^2} + N_\psi \frac{d\psi}{dt} + R_\psi \psi &= M_\psi \sin(\omega_e t - \epsilon_\psi) \\ I_\varphi \frac{d^2\varphi}{dt^2} + N_\varphi \frac{d\varphi}{dt} + R_\varphi \varphi &= M_\varphi \sin(\omega_e t - \epsilon_\varphi) \end{aligned}\right\} \quad \text{(a-1)}$$

In these equations the relative motion of ship and sea is neglected. Justification for this neglect lies in the intent to set forth more clearly the theory presented in this paper.

Some remarks will now be made on the derivation of the hydrodynamic coefficients entering into the equations of motion.

The Inertial Reactions

The coefficients of the acceleration terms represent either the virtual mass or the virtual moment of inertia of the vessel about the axis of motion.

Heave. The virtual mass in heave is written

$$M_z \triangleq \rho\nabla(1 + k_z) \qquad \text{(a-2)}$$

where the linear inertia coefficient, k_z, represents the ratio of the hydrodynamic mass to that of the vessel. This was shown in reference [9] to be expressible as

$$k_z \triangleq \frac{\pi L B^2}{8\nabla} \cdot (C_l)_z C_z \qquad \text{(a-3)}$$

by relating it to the cylinder of semi-circular section whose radius equals the half-breadth of the ship. In this equation $(C_l)_z$ is an aspect ratio correction. It is obtained as the ratio of the linear inertia coefficient of the semi-ellipsoid having the same principal proportions as the ship to that of the semi-elliptic cylinder of section equal to the maximum section of the ellipsoid.

C_z is an averaged hydrodynamic mass coefficient defined as

$$C_z \triangleq \frac{4}{LB^2} \int_{-L/2}^{L/2} Y_0^2 C_z(X)\, dX = \frac{1}{2} \int_{-1}^{1} \eta^2 C_z(\xi)\, d\xi \qquad \text{(a-4)}$$

where $Y_0 \equiv \eta B/2$ is the water line half-breadth of the section. The coefficient $C_z(X) \equiv C_z(\xi)$ at a section is obtained as a function of its beam-draft ratio $2Y_0/Z_0 \equiv \eta B/\zeta H$ and its sectional area coefficient $\beta(X) \equiv \beta(\xi)$ as shown by Prohaska [17].

Pitch. The virtual moment of inertia in pitch is written

$$I_\psi \triangleq \rho I_y (1 + k_{yy}) \qquad \text{(a-5)}$$

where the angular inertia coefficient, k_{yy}, represents the ratio of the hydrodynamic moment of inertia to the moment of inertia of the vessel, both taken with reference to the axis of pitch (Y-Y). This was shown to be expressible as

$$k_{yy} \triangleq \frac{\pi L^3 B^2}{32 I_y} \cdot (C_l)_\psi C_\psi \qquad \text{(a-6)}$$

on the basis of a two-dimensional strip theory in which the semi-circular cylinder is again used as a basis of reference. In this equation the correction for aspect ratio $(C_l)_\psi$ is obtained as the ratio of the angular inertia coefficient in pitch of the semi-ellipsoid and semi-elliptic cylinder introduced when discussing heave.

C_ψ is an averaged hydrodynamic moment of inertia coefficient defined as

$$C_\psi \triangleq \frac{16}{L^3 B^2} \int_{-L/2}^{L/2} Y_0^2 X^2 C_z(X)\, dX = \frac{1}{2} \int_{-1}^{1} \eta^2 \xi^2 C_z(\xi)\, d\xi \qquad \text{(a-7)}$$

Roll. The virtual moment of inertia in roll is written

$$I_\varphi \triangleq \rho I_x (1 + k_{xx}) \qquad \text{(a-8)}$$

where the angular inertia coefficient k_{xx} represents the ratio of the hydrodynamic moment of inertia to the moment of inertia of the vessel both taken with reference to the axis of roll (X-X). In the derivation that follows the inertia coefficient is referred to that obtaining for a plate whose breadth equals that of the section. The reason that the circle can no longer be used as reference is that its hydrodynamic moment of inertia is zero. For a plate of width $2Y_0$ floating at the surface the hydrodynamic moment of inertia is given by $\frac{1}{16}\pi\rho Y_0^4$. For a ship section of half-breadth Y_0 and draft Z_0 the hydrodynamic moment of inertia is written

$$C_\varphi \cdot \frac{\pi\rho Y_0^4}{16} = C_\varphi \cdot \frac{\pi\rho B^4 \eta^4}{256} \qquad \text{(a-9)}$$

where the coefficient C_φ is a function of geometric properties of the section. This coefficient has been evaluated only for some simple geometric forms (ellipse, tetragon, octagon), a tabulation of which was given by Wendel [25]. In contrast to the case for heave and pitch, the extension to ship forms has not yet been made. For convex sections the coefficient can be obtained by interpolation between the value corresponding to a semi-ellipse and that for a rectangle of equal proportions.

As an aid to this interpolation curves of C_φ for the rectangle and the semi-ellipse are plotted in Fig. A-1. The values of C_φ for the semi-ellipse can be readily obtained from the formula

$$C_\varphi = (1 - Z_0^2/Y_0^2)^2 \qquad \text{(a-10)}$$

Solutions for C_φ applicable to concave sections are still wanting. It appears that for such sections C_φ decreases with increasing section coefficient and that an empirical formula like

$$C_\varphi \cong 0.74 - 130\beta^8(X) \qquad \text{(a-11)}$$

gives a reasonable first approximation within the important range $0.3 < \beta(X) < 0.5$.

Although bilge keels contribute but a negligible effect in heave and pitch, they need to be taken into account in roll. But again one is forced into empiricism by the lack of a convenient procedure for rational design. Wendel has given values of C_φ

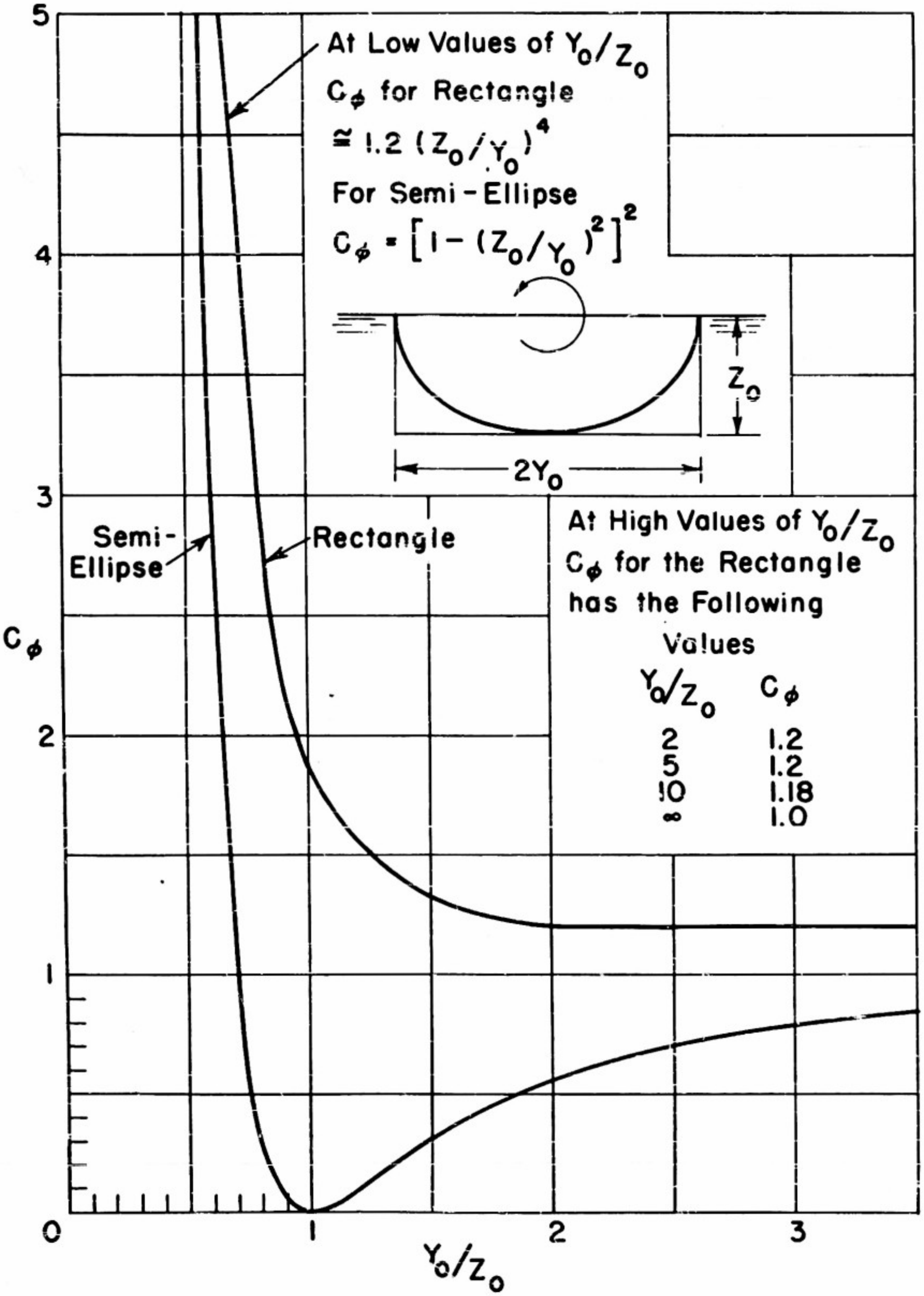

FIG. A-1.—$C\phi$ FOR RECTANGLE AND SEMI-ELLIPSE

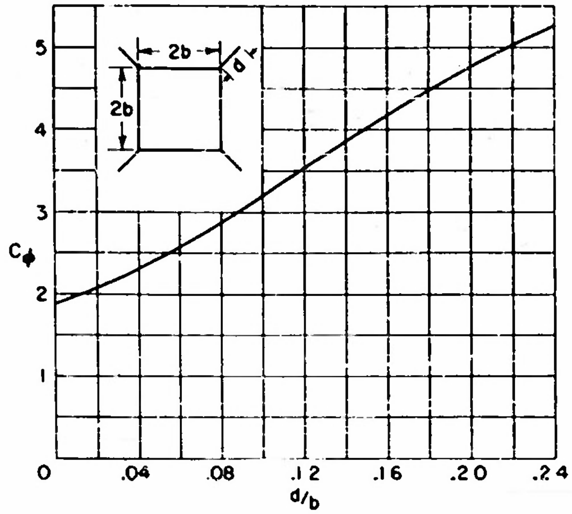

FIG. A-2.—EFFECTS OF DIAGONAL FINS ON $C\phi$ (WENDEL)

for a square section with diagonal fins at the corners, Fig. *A*-2. The effect of the bilge keels is large and over a wide range amounts to a 7.7% increase in C_φ for each 1% increase in the ratio of bilge keel to half-breadth. For want of data directly applicable to ship sections, the numerical increase as given by this figure should be used.

The hydrodynamic moment of inertia is obtained by integration

$$\rho I_{xx} = \frac{\pi\rho}{16}\int_{-L/2}^{L/2} C_\varphi(X)\, Y_0^4\, dX \equiv \frac{\pi\rho LB^4}{512}\int_{-1}^{1} C_\varphi(\xi)\eta^4\, d\xi \qquad \text{(a-12)}$$

where $C_\varphi(X) \equiv C_\varphi(\xi)$ includes a correction for the effect of bilge keels. The weighted average coefficient C_φ is given by

$$C_\varphi \triangleq \frac{16}{LB^4}\int_{-L/2}^{L/2} C_\varphi(X)\, Y_0^4\, dX \equiv \frac{1}{2}\int_{-1}^{1} C_\varphi(\xi)\eta^4\, d\xi \qquad \text{(a-13)}$$

By reference to the moment of inertia of the ship about her longitudinal-axis, I_x, the angular inertia coefficient in roll then results as

$$k_{xx} \triangleq \frac{\pi LB^4}{256 I_x}\cdot (C_i)_\varphi C_\varphi \qquad \text{(a-14)}$$

Where again $(C_i)_\varphi$ is an aspect ratio correction obtained as the ratio of the angular inertia coefficient in roll of the semi-ellipsoid and semi-elliptic cylinder introduced when discussing heave.

THE DAMPING REACTIONS

The coefficients of the velocity terms, or damping coefficients, have already been derived for heave and pitch [20]. They will be recalled here in the simplified form that neglects the relative wave motion and modified slightly so as to be more consistent with subsequent derivations of the remaining coefficients.

Heave. The damping coefficient is given by

$$N_z \triangleq \int_{-L/2}^{L/2} N_z(X)\, dX \equiv \frac{L}{2}\int_{-1}^{1} N_z(\xi)\, d\xi \qquad \text{(a-15)}$$

where $N_z(X) \equiv N_z(\xi)$ is a sectional damping coefficient defined by

$$N_z(X) \triangleq \frac{4\rho\omega_e \sin k_0 Y_0}{k^2}\cdot e^{k[\bar{Z}(X)-h]} \qquad \text{(a-16)}$$

and $[\bar{Z}(X) - h]$ represents the mean draft of the section. By introducing a mean draft for the whole vessel $(\bar{Z} - h) \cong -H\sigma$, where σ is the vertical coefficient, the damping coefficient is approximated by

$$N_z = \frac{4\rho\omega_e}{k^2}\cdot e^{-2kH\sigma}\int_{-L/2}^{L/2} \sin^2 kY_0\, dX \qquad \text{(a-17)}$$

For use of calculation it is convenient to express $\sin^2 kY_0$ in terms of kY_0. To this effect write

$$\sin^2 kY_0 \triangleq s^2(X)\cdot kY_0 \qquad \text{(a-18)}$$

where

$$s^2(X) \triangleq \frac{\int_0^{Y_0} \sin^2 kY_0\, dY}{\int_0^{Y_0} k^2 Y_0^2} = \frac{3}{2k^2Y_0^2}\left(1 - \frac{\sin 2kY_0}{2kY_0}\right) \triangleq s^2(\xi) \qquad \text{(a-19)}$$

This coefficient is plotted in Fig. *A*-3. With this substitution the damping coefficient in heave is written

$$N_z = 4\rho\omega_e\, e^{-2kH\sigma}\int_{-L/2}^{L/2} s^2(X)\, Y_0^2\, dX \equiv \tfrac{1}{2}LB^2\rho\omega_e e^{-2kH\sigma}\int_{-1}^{1} s^2(\xi)\eta^2\, d\xi \qquad \text{(a-20)}$$

Pitch. The damping coefficient is given by

$$N_\psi \triangleq \int_{-L/2}^{L/2} N_z(X)X^2\, dX \equiv \frac{L^3}{8}\int_{-1}^{1} N_z(\xi)\xi^2\, d\xi \qquad \text{(a-21)}$$

Following the same reasoning as for heave it can be easily verified that

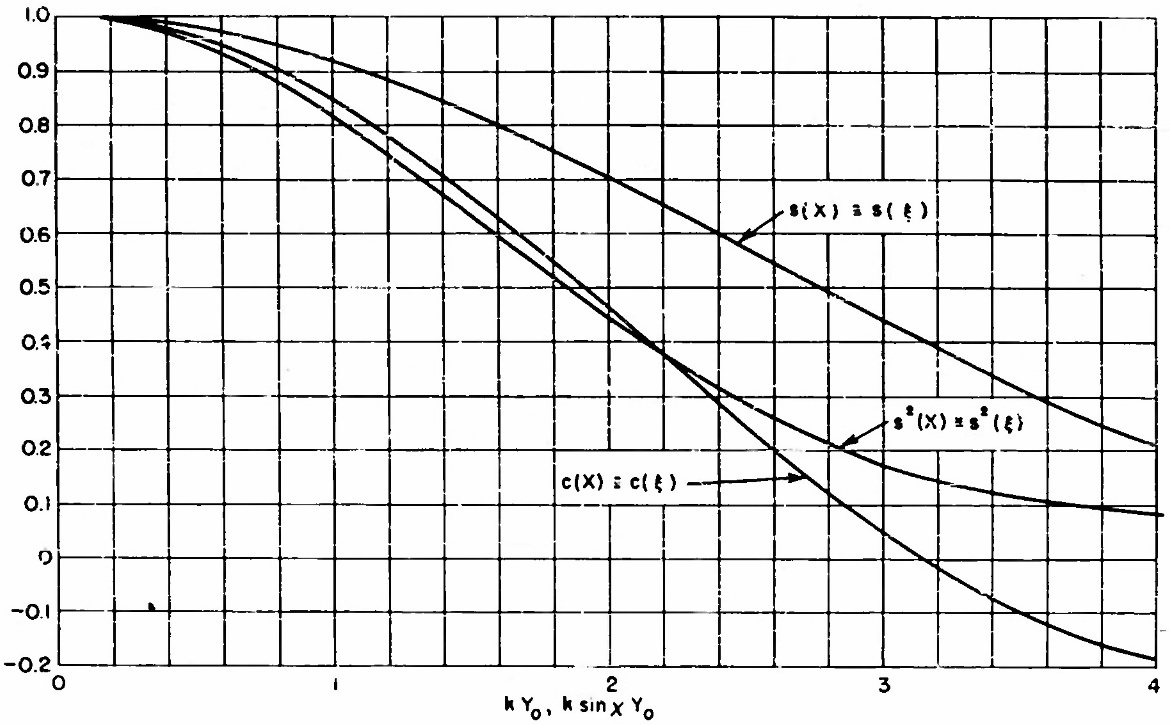

FIG. A-3.—LINEARIZING COEFFICIENTS

$$N_\psi = 4\rho\omega_e e^{-2kH_e}\int_{-L/2}^{L/2} s^2(X)X^2Y_0^2\,dX$$

$$\equiv \tfrac{1}{8}L^3B^2\rho\omega_e e^{-2kH_e}\int_{-1}^{1} s^2(\xi)\xi^2\eta^2\,d\xi \qquad \text{(a-22)}$$

Roll. The damping reaction in roll consists of two components: The first is similar in nature to the damping reactions in heave and pitch, arises through the generation of waves, varies linearly with velocity and can be treated analytically. The second arises through surface friction and flow separation, varies with the square of the velocity, can be treated only empirically and must be linearized to be introduced in the equation of motion.

Damping from wave generation (or potential damping) will be discussed first. To this end consider a pulsating source located below the surface at a depth $(Z - h)$. If the strength of the source per unit length is $q \cos \omega_e t$, then the elevation at a point $Y = b$ of the regular surface waves propagated along the positive Y axis is

$$r_m = \frac{2\pi q\omega_e}{g}\cdot e^{k(Z-h)}\cos(\omega_e t - kb) \qquad \text{(a-23)}$$

and the rate of energy dissipation to one side is

$$\frac{dE}{dt} = \frac{\rho g^2 r_m^2}{4\omega_e}$$

$$= \pi^2\rho q^2\omega_e e^{2k(Z-h)} \qquad \text{(a-24)}$$

Apply the foregoing to an arbitrary section of halfbreadth $Y_0 = \tfrac{1}{2}\eta B$ and draft $Z_0 = \zeta H$ performing oscillations in roll.

$$\varphi = \varphi_m \cos \omega_e t \qquad \text{(a-25)}$$

The motion is assumed to be two-dimensional and due to a distribution of sources of varying strength over the section. The strength of a source at a point (Y_s, Z_s) of the hull surface is taken to be

$$dq = \frac{1}{2\pi}\cdot\frac{ds}{dt}\cdot dl \qquad \text{(a-26)}$$

where s is the total displacement of the point when the vessel rolls about her center of gravity through an angle φ, i.e.,

$$ds = \varphi\sqrt{Y_s^2 + Z_s^2}\cdot dl \qquad \text{(a-27)}$$

and dl is an element of girth, see Fig. A-4. The strength of the source, dq, can be written more conveniently in terms of the vertical and horizontal components of the girth, $m\ dl$ and $n\ dl$.

If these are introduced

$$ds = \varphi(Z_s m\, dl - Y_s n\, dl) \quad \text{(a-28)}$$

and

$$dq = \frac{1}{2\pi}(mZ_s - nY_s)\frac{d\varphi}{dt}\cdot dl \quad \text{(a-29)}$$

from which

$$dq = -1/2\pi(mZ_s - nY_s)\omega_e\varphi_m \sin \omega_e t\cdot dl \quad \text{(a-30)}$$

The amplitude of the resulting surface wave at point $Y = b$ is

$$dr_m = \frac{2\pi\omega_e}{g}\cdot dq_m \cdot e^{k(Z-h)} \cos [\omega_e t - k(b - Y)]$$

$$= -\frac{\omega_e^2\varphi_m}{g}(mZ_s - nY_s)e^{k(Z-h)} \cos [\omega_e t - k(b - Y)]\, dl \quad \text{(a-31)}$$

where Y is the ordinate of the source. If the integration is made along the contour the resulting surface wave is

$$r_m = -\frac{\omega_e^2\varphi_m}{g}\oint e^{k(Z-h)}(mZ_s - nY_s) \cos [\omega_e t - k(b - Y)]dl \quad \text{(a-32)}$$

The integration is readily carried on the assumption that only the sources in the positive half of the vessel account for the waves generated in the positive direction. The contour integral is replaced by linear integrals by introducing $m\, dl = dZ$ and $n\, dl = dY$. This results in the following expansion

$$r_m = -\frac{\omega_e^2\varphi_m}{g}\left\{\cos(\omega_e t - kb)\left[\int_{h+H}^{h} e^{k(Z-h)} Z_s \cos kY_s\, dZ + \int_0^{B/2} e^{k(Z-h)} Y_s \cos kY_s\, dY\right]\right.$$
$$\left. - \sin(\omega_e t - kb)\left[\int_{h+H}^{h} e^{k(Z-h)} Z_s \sin kY_s\, dZ + \int_0^{B/2} e^{k(Z-h)} Y_s \sin kY_s\, dY\right]\right\} \quad \text{(a-33)}$$

These integrals are evaluated by assuming average values for the trigonometric and exponential terms as follows:

(a) Replace the term $\cos kY_s$ in the integrand by its average value and remove this from under the integral sign

$$c(X) \triangleq \frac{1}{Y_0}\int_0^{Y_0} \cos kY_s\, dY \quad \text{(a-34)}$$

(b) Replace similarly the term $\sin kY_s$ by its average

$$(kY_0)\, s(X) \triangleq kY_0 \frac{\int_0^{Y_0} \sin kY_s\, dY}{\int^{Y_0} kY_s\, dY} \quad \text{(a-35)}$$

Both factors $c(X)$ and $s(X)$ are plotted in Fig. A-3. Two average values of $e^{k(Z-h)}$ are obtained depending upon whether the integration is carried out vertically or horizontally.

(c) The average value of the exponential term integrated vertically from $Z = Z_0$ to $Z = h$ is simply

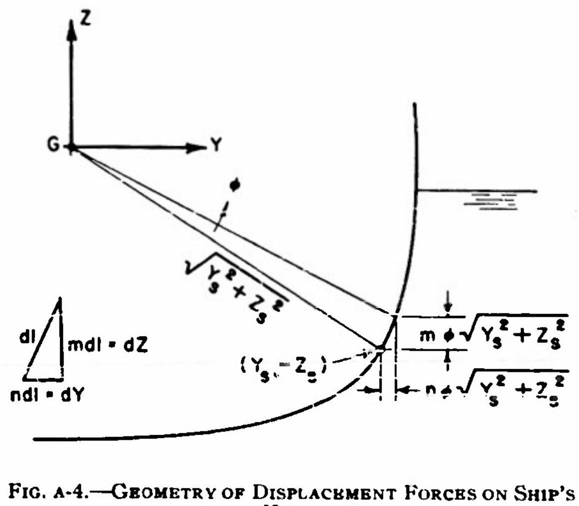

FIG. A-4.—GEOMETRY OF DISPLACEMENT FORCES ON SHIP'S HULL

$$e_Z^{(kZ)} \triangleq \frac{1}{Z_0 - h} \int_{Z_0}^{h} e^{k(Z-h)}\, dZ$$

$$= -\frac{1}{k(Z_0 - h)} [1 - e^{k(Z_0-h)}] \quad \text{(a-36)}$$

(d) At a section the average value of the exponential term integrated horizontally is

$$e_Y^{(kZ)} \triangleq e^{kZ_0 B(X)} \quad \text{(a-37)}$$

If these substitutions are made in equation (a-33) and the integrations are carried out, the following expression results for the surface waves

$$r_m = -\frac{\omega_e^2 \varphi_m}{2g} \Big\{ \{e_Z^{(kZ)} [(Z_0 - h)^2 - 2h(Z_0 - h)] - e_Y^{(kZ)} Y_0^2\} \cdot [c(X) \cos(\omega_e t - kb) - s(X) k Y_0 \sin(\omega_e t - kb)] \Big\} \quad \text{(a-38)}$$

The amplitude of this wave is

$$r_m = \frac{\omega_e^2 \varphi_m}{2g} \Big\{ \{e_Z^{(kZ)} [(Z_0 - h)^2 - 2h(Z_0 - h)] - e_Y^{(kZ)} Y_0^2\} \cdot \sqrt{c^2(X) + s^2(X) k^2 Y_0^2} \Big\} \quad \text{(a-39)}$$

Thus the average rate of energy dissipation to one side is

$$\frac{dE}{dt} = \frac{\rho \omega_e^3 \varphi_m^2}{16} \Big\{ \{e_Z^{(kZ)} [(Z_0 - h)^2 - 2h(Z_0 - h)] - e_Y^{(kZ)} Y_0^2\}^2 \cdot [c^2(X) + s^2(X) k^2 Y_0^2] \Big\} \quad \text{(a-40)}$$

If this is equated to the average rate of work performed by the damping moment $(d\varphi/dt)\,[N_\varphi(X)]_W$ which is

$$[N_\varphi(X)]_W \left(\frac{d\varphi}{dt}\right)^2 = \frac{1}{2} N_\varphi(X)\, \omega_e^2 \varphi_m^2 = \frac{dE}{dt} \quad \text{(a-41)}$$

the sectional damping coefficient result as

$$[N_\varphi(X)]_W = \frac{1}{8} \rho \omega_e \Big\{ \{e_Z^{(kZ)} [(Z_0 - h)^2 - 2h(Z_0 - h)] - e_Y^{(kZ)} Y_0^2\}^2 \cdot [c^2(X) \times s^2(X) k^2 Y_0^2] \Big\} \quad \text{(a-42)}$$

The potential damping coefficient in roll is thus

$$(N_\varphi)_W \triangleq \int_{-L/2}^{L/2} [N_\varphi(X)]_W\, dX$$
$$\equiv \tfrac{1}{2} L \int_{-1}^{1} [N_\varphi(\xi)]_W\, d\xi \quad \text{(a-43)}$$

It is convenient to average the exponential terms for the whole vessel. Referring these averages to the draft H in lieu of $-(Z_0 - h)$ and to the beam B, there results

$$e_Z^{(kZ)} = \frac{1}{kH} [1 - e^{-kH}] \quad \text{(a-44)}$$

$$e_Y^{(kZ)} = e^{-kH\sigma} \quad \text{(a-45)}$$

where σ is the vertical coefficient. Consequently

$$(N_\varphi)_W = \frac{1}{8} \rho \omega_e \Big\{ \frac{1}{2} \cdot L(H - 2h) \cdot \frac{1}{k} \cdot (1 - e^{-kH}) \int_{-1}^{1} c^2(\xi)\, d\xi$$
$$+ \frac{1}{8} \cdot LB^2 H(H - 2h) k^2 (1 - e^{-kH}) \int_{-1}^{1} \eta^2 s^2(\xi)\, d\xi$$
$$- \frac{1}{8} LB^2 e^{-kH\sigma} \int_{-1}^{1} \eta^2 c^2(\xi)\, d\xi - \frac{1}{32} LB^4 k^2 e^{-kH\sigma} \int_{-1}^{1} \eta^4 s^2(\xi)\, d\xi \Big\} \quad \text{(a-46)}$$

Viscous damping arising from surface friction and flow separation will now be discussed. The frictional component will be considered first. The frictional resistance of an element of hull surface $dl\,dX$ located at the point (X_s, Y_s, Z_s) is

$$dF = \frac{1}{2}\rho c_f (Y_s^2 + Z_s^2)\left(\frac{d\varphi}{dt}\right)^2 dl\,dX \quad \text{(a-47)}$$

where c_f is an empirically derived coefficient which equals approximately 0.02. If this force is resolved into horizontal and vertical components $m\,dF$ and $n\,dF$, its moment about the origin is

$$dM = \frac{1}{2}\rho c_f \left(\frac{d\varphi}{dt}\right)^2 (Y_s^2 + Z_s^2)(mY + nZ)\,dl\,dX$$
$$= \frac{1}{2}\rho c_f \left(\frac{d\varphi}{dt}\right)^2 (Y_s^2 + Z_s^2)(Y\,dZ + Z\,dY)\,dX \quad \text{(a-48)}$$

since $m\,dl = dZ$ and $n\,dl = dY$. The total moment about the origin of this force obtained by integration results as

$$M = 2\rho c_f J_x \left(\frac{d\varphi}{dt}\right)^2 \quad \text{(a-49)}$$

where J_x is the moment of inertia of the volume of displacement about the longitudinal axis.

The corresponding damping coefficient is

$$(N_\varphi)_f = 2\rho c_f J_x \quad \text{(a-50)}$$

The component of viscous damping arising from flow separation is derived in a manner similar to the frictional component. It will be assumed here that flow separation occurs only as a consequence of bilge keels of constant width w placed normally to the hull surface along the trace (Y_B, Z_B). The separation resistance of an element of bilge keel of length dX is

$$dF = \frac{1}{2}\rho c_s w \left(\frac{d\varphi}{dt}\right)^2 (Y_B^2 + Z_B^2)\,dX \quad \text{(a-51)}$$

where c_s is an empirically derived coefficient which equals approximately 1.6. Its moment about the origin is

$$dM = \frac{1}{2}\rho c_s w \left(\frac{d\varphi}{dt}\right)^2 (Y_B^2 + Z_B^2)(mY_B + nZ_B)\,dX \quad \text{(a-52)}$$

from which the total moment (both sides) results as

$$M = \rho c_s w \left(\frac{d\varphi}{dt}\right)^2 \int (Y_B^2 + Z_B^2) \cdot (mY_B + nZ_B)\,dX \quad \text{(a-53)}$$

the limits of integration being the extremities of the bilge keels. The corresponding damping coefficient is then

$$(N_\varphi)_s = \rho c_s w \int (Y_B^2 + Z_B^2)(mY_B + nZ_B)\,dX \quad \text{(a-54)}$$

The viscous damping reaction

$$(N_\varphi)_v \left(\frac{d\varphi}{dt}\right)^2 = [(N_\varphi)_f + (N_\varphi)_s]\left(\frac{d\varphi}{dt}\right)^2 \quad \text{(a-55)}$$

is thus a quadratic function of the velocity. It cannot be introduced directly in the equation of motion, but needs first be linearized. As shown by Braehmig [3], the linearization consists in finding an equivalent viscous damping coefficient which depends on the first power of the velocity and whose magnitude is determined by the condition that the work absorbed during a cycle remains unaltered. Then, dropping obvious subscripts for the moment,

$$4N_1 \int_0^{T/4} \left(\frac{d\varphi}{dt}\right)\cdot\left(\frac{d\varphi}{dt}\right) dt = 4N_2 \int_0^{T/4} \left(\frac{d\varphi}{dt}\right)^2 \cdot \left(\frac{d\varphi}{dt}\right) dt \quad \text{(a-56)}$$

If the plausible assumption is made that such change in the damping reaction will have but a negligible effect on the motion

$$\varphi = \varphi_m \sin(\omega_e t - \epsilon_\varphi) \quad \text{(a-57)}$$

the condition is obtained that

$$N_1 \int_0^{T/4} \cos^2(\omega_e t - \epsilon_\varphi)\,dt = N_2 \int_0^{T/4} \cos^3(\omega_e t - \epsilon_\varphi)\,dt \quad \text{(a-58)}$$

from which

$$N_1 = \frac{8}{3\pi} N_2 \quad \text{(a-59)}$$

and

$$(N_\varphi)_v \left(\frac{d\varphi}{dt}\right)^2 = \frac{8}{3\pi}(N_\varphi)_v \left(\frac{d\varphi}{dt}\right) \quad \text{(a-60)}$$

The total damping reaction is thus

$$N_\varphi \left(\frac{d\varphi}{dt}\right) = \left[(N_\varphi)_w + \frac{8}{3\pi}(N_\varphi)_v\right]\left(\frac{d\varphi}{dt}\right) \quad \text{(a-61)}$$

The Restoring Reactions

The third terms in the equations of motion, (a-1), represent restorations, i.e., hydrostatic reactions which take place when the vessel is disturbed from her position of equilibrium in calm water. These coefficients are, of course, well known. But since the derivation of the excitation coefficients is of greater complexity yet follows a parallel procedure, it is believed con-

venient to outline briefly the derivation of the coefficients of restoration. In the discussion that follows the restorations in heave, pitch and roll will be derived simultaneously.

Consider a vessel displaced from her position of equilibrium by a motion which combines heave, pitch and roll. As a result of this complex motion, the waterline for the ship at rest, WL in Fig. *A*-5, moves to a new position W_1L_1 because of linear displacements along the vertical (Z) axis and an angular displacement about the origin. The vertical displacement of the waterline is the resultant of the linear components of heave (z) and pitch (ψ). The angular displacement is simply the roll angle φ.

When the vessel is in a position of static equilibrium the hydrostatic force per unit area acting normal to the hull surface at the point (X_s, Y_s, $-Z_s$) is

$$dF = \rho g(-Z_s + h) \qquad \text{(a-62)}$$

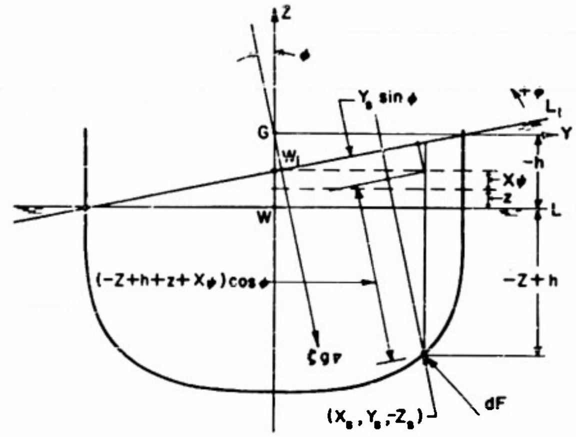

FIG. A-5. GEOMETRY OF RESTORATION

When the vessel is disturbed from her position of equilibrium the normal hydrostatic force becomes

$$dF = \rho g\,[(-Z_s + h + z + X_s\psi)\cos\varphi + Y_s\sin\varphi] \qquad \text{(a-63)}$$

For small angular displacements the linearization $\cos\varphi \cong 1$ and $\sin\varphi \cong \varphi$ is introduced giving a change in hydrostatic force equal to

$$\rho g(z + X_s\psi + Y_s\varphi) \qquad \text{(a-64)}$$

the three terms in the parentheses being due, respectively, to heave, pitch and roll.

To obtain the restoring reactions this force and the moments it gives rise to will be integrated over the hull surface. To this end it is convenient to resolve the force into horizontal and vertical components, $m\ dF$ and $n\ dF$. Consider first the integral of the vertical component which is the restoration in heave

$$nF = \rho g \int_{-L/2}^{L/2} \oint n(z + X_s\psi + Y_s\varphi)\,dl\,dX \qquad \text{(a-65)}$$

where dl is an element of girth. Because of antisymmetry along the X and Y axes

$$\int_{-L/2}^{L/2} \oint nX_s\psi\,dl\,dX = \int_{-L/2}^{L/2} \oint nY_s\varphi\,dl\,dX \equiv 0 \qquad \text{(a-66)}$$

The terms in ψ and φ are consequently dropped. Introducing $n\,dl = dY$

$$nF = \rho g \int_{-L/2}^{L/2} \int z\,dY\,dX = \rho g A(Z) z \qquad \text{(a-67)}$$

where $A(Z)$ is the waterplane area.

Consider secondly the integral of the horizontal component which is a swaying force associated with roll.

$$mF = \rho g \int_{-L/2}^{L/2} \oint m(z + X_s\psi + Y_s\varphi)\,dl\,dX \qquad \text{(a-68)}$$

Because of antisymmetry along the X and Y axes

$$\int_{-L/2}^{L/2} \oint mz\,dl\,dX = \int_{-L/2}^{L/2} \oint mX_s\psi\,dl\,dX \equiv 0 \qquad \text{(a-69)}$$

The terms in z and ψ are consequently dropped. Introducing $m\,dl = dZ$

$$mF = \rho g\varphi \int_{-L/2}^{L/2} \int Y_s dZ\,dX = \rho g\Delta\varphi \qquad \text{(a-70)}$$

Consider thirdly the component of the righting moment in roll due to the integral of the horizontal force.

$$M = -\rho g \int_{-L/2}^{L/2} \oint mZ(z + X_s\psi + Y_s\varphi)\,dl\,dX \qquad \text{(a-71)}$$

Because of antisymmetry along the Y axis

$$\int_{-L/2}^{L/2} \oint mZz\,dl\,dX = \int_{-L/2}^{L/2} \oint mX_sZ\psi\,dl\,dX \equiv 0 \qquad \text{(a-72)}$$

The terms in z and ψ are consequently dropped. With $m\,dl = dZ$

$$M = -\rho g\varphi \int_{-L/2}^{L/2} \int Y_s Z\,dZ\,dX = -\rho g\nabla\bar{Z}\varphi \qquad \text{(a-73)}$$

where $\bar{Z}$ is the vertical lever arm to the centroid of the immersed volume, i.e., center of buoyancy.

Consider fourthly, the component of the righting moment in roll due to the integral of the vertical force

$$M = \rho g \int_{-L/2}^{L/2} \oint nY(z + X_s\psi + Y_s\varphi)\, dl\, dX \qquad \text{(a-74)}$$

By antisymmetry along the Y axis

$$\int_{-L/2}^{L/2} \oint nY_s z\, dl\, dX = \int_{-L/2}^{L/2} \oint nYX_s\psi\, dl\, dX \equiv 0 \qquad \text{(a-75)}$$

The terms in z and ψ are consequently dropped. With $n\, dl = dY$

$$M = \rho g\varphi \int_{-L/2}^{L/2} \int_{-Y_s}^{Y_s} Y_s^2\, dY = \rho g J_x\, \varphi \qquad \text{(a-76)}$$

where J_x is the moment of inertia of the waterplane about the longitudinal axis.

Since the horizontal shift of the center of buoyancy $= \bar{Y} = J_x\varphi/\nabla$, this moment is equivalent to that caused by a vertical force $\rho g\nabla$ located at the displaced center of buoyancy, see Fig. *A*-6.

The total righting moment in roll is best expressed by obtaining the resultant at the displaced center of buoyancy of the horizontal and vertical forces giving rise to the righting moment in roll. Since these are, respectively, $\varphi\rho g\nabla$ and $\rho g\nabla$, the resultant, which is the buoyant force, is $\rho g\nabla$. This acts at an angle φ to the original vertical and intersects it at the metacenter.

The righting moment is then the well-known expression

$$M = \rho g\varphi\,(J_x - \nabla\bar{Z}) = \rho g\nabla\cdot\overline{GM}\cdot\varphi \qquad \text{(a-77)}$$

Consider fifthly the righting moment in pitch. This is due to the vertical force and is

$$M = \rho g \int_{-L/2}^{L/2} \oint nX_s(z + X_s\psi + Y_s\varphi)\, dl\, dX \qquad \text{(a-78)}$$

By antisymmetry along the X and Y axes

$$\int_{-L/2}^{L/2} \oint nX_s z\, dl\, dX = \int_{-L/2}^{L/2} \oint nXY_s\varphi\, dl\, dX \equiv 0 \qquad \text{(a-79)}$$

The terms in z and φ are consequently dropped. With $n\, dl = dY$

$$M = \rho g\psi \int_{-L/2}^{L/2} \int_{-Y_s}^{Y_s} X_s^2\, dY\, dX = \rho g J_y \psi \qquad \text{(a-80)}$$

where J_y is the moment of inertia of the waterplane about the Y axis.

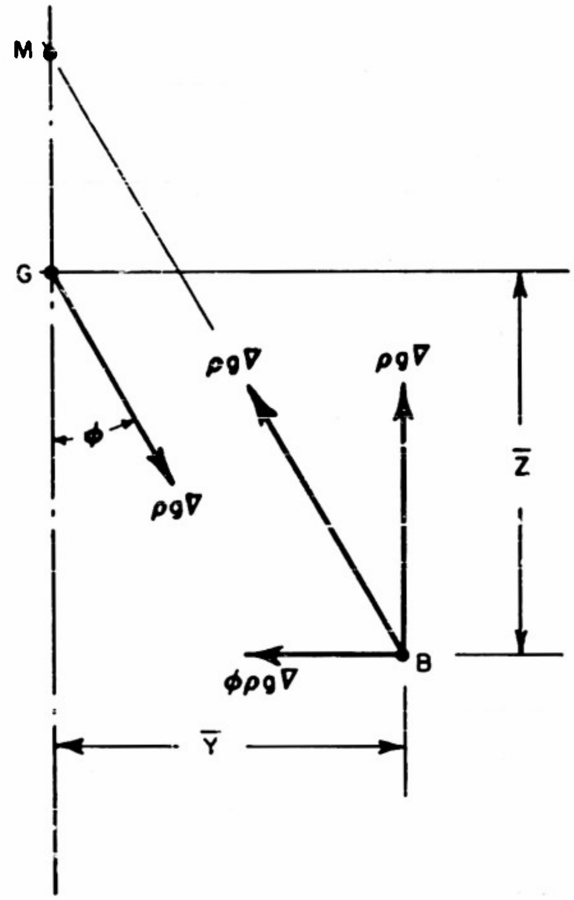

Fig. A-6.—Restoration in Roll

The Excitation

The right-hand terms in the equations of motion represent excitations, i.e., hydrodynamic forces and moments which occur when a vessel progresses in a seaway composed of regular waves retaining the attitude corresponding to equilibrium in calm water.

Consider a vessel heading at an angle χ into a regular seaway, Figs. *A*-7 and *A*-8. The wave ordinate at a point X, Y of the surface is

$$r = r_m \cos\,[(k\cos\chi)X + (k\sin\chi)Y - \omega t + \epsilon)] \qquad \text{(a-81)}$$

where $k \triangleq 2\pi/\lambda$. The hydrodynamic head at the point $(X, Y, -Z)$ is the wave ordinate multiplied by $e^{k(Z-h)}$. The corresponding hydrodynamic force per unit area acting normal to the hull surface at the point $(X_s, Y_s, -Z_s)$ is

$$dF = \rho g r_m e^{k(Z-h)} \cos\,[(k\cos\chi)X_s + (k\sin\chi)Y_s - \omega_e t + \epsilon] \qquad \text{(a-82)}$$

To obtain the excitations this force and the moments it gives rise to will be integrated over the hull surface. Again it will be convenient to resolve the force into horizontal and vertical components $m\,dF$ and $n\,dF$.

Consider first the integral of the vertical component, which is the excitation in heave.

$$F_z = nF = \rho g r_m \int_{-L/2}^{L/2} \oint n e^{k(Z-h)} \cdot \cos\left[(k\cos\chi)X_s + (k\sin\chi)Y_s - \omega t + \epsilon\right] dl\, dX \quad \text{(a-83)}$$

The contour integral is expanded into products of time dependent and time invariant factors

$$\cos(\omega_e t - \epsilon) \oint n e^{k(Z-h)} \cos\left[(k\cos\chi)X_s + (k\sin\chi)Y_s\right] dl + \sin(\omega_e t - \epsilon) \oint n e^{k(Z-h)} \sin\left[(k\cos\chi)X_s + (k\sin\chi)Y_s\right] dl \quad \text{(a-84)}$$

By antisymmetry the first term reduces to

$$2\cos(\omega_e t - \epsilon)\cos(k\cos\chi)X_s \oint n e^{k(Z-h)} \cos(k\sin\chi)Y_s\, dl = 2\cos(\omega_e t - \epsilon)\cos(k\cos\chi)X_s \int e^{k(Z-h)} \cos(k\sin\chi)Y_s\, dY \quad \text{(a-85)}$$

To linearize the solution proceed as for damping, replacing the trigonometric and exponential terms by their average values at a section. Thus for $\int \cos(k\sin\chi)Y_s\, dY$ substitute

$$c(X) \triangleq \frac{1}{Y_0}\int_0^{Y_s} \cos(k\sin\chi)Y_s\, dY \doteq c(\xi) \quad \text{(a-86)}$$

A plot of this coefficient is given in Fig. *A*-3. For the exponential term use the expression of equation (a-32), obtaining

$$2\cos(\omega_e t - \epsilon)e_z^{(kZ)}\, c(X)\, Y_0 \cos(k\cos\chi)X \quad \text{(a-87)}$$

Integrating this expression along the length and introducing the dimensionless parameters

$$\eta \triangleq 2Y_0/B,\ \xi \triangleq 2X/L,\ \gamma \triangleq \pi L/\lambda, \quad \text{(a-88)}$$

the first term of equation (a-84) results as

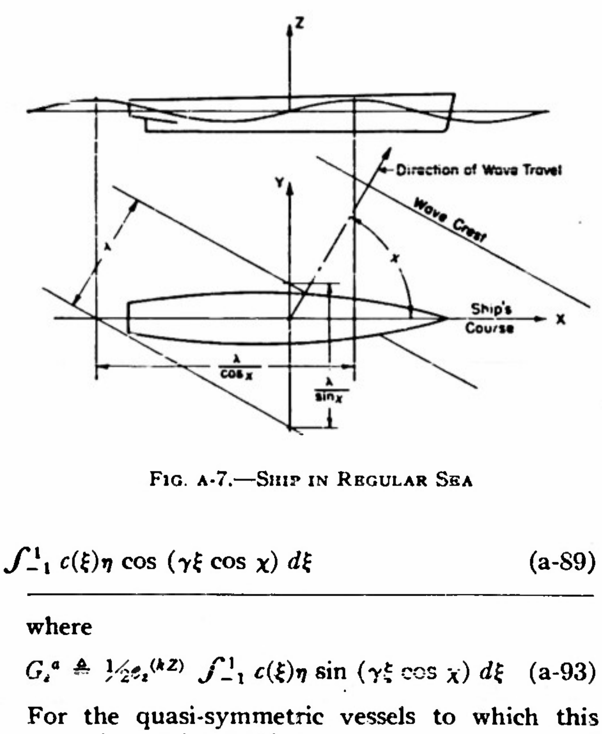

FIG. A-7.—SHIP IN REGULAR SEA

$$\tfrac{1}{2}\cos(\omega_e t - \epsilon)\, e_z^{(kZ)}\, LB \int_{-1}^{1} c(\xi)\eta\cos(\gamma\xi\cos\chi)\, d\xi \quad \text{(a-89)}$$

Defining

$$G_z' \triangleq \tfrac{1}{2}e_z^{(kZ)} \int_{-1}^{1} c(\xi)\eta\cos(\gamma\xi\cos\chi)\, d\xi \quad \text{(a-90)}$$

the first term becomes

$$LBG_z'\cos(\omega_e t - \epsilon) \quad \text{(a-91)}$$

The second term of equation (a-84) reduces similarly to the expression

$$LBG_z^a\sin(\omega_e t - \epsilon) \quad \text{(a-92)}$$

where

$$G_z^a \triangleq \tfrac{1}{2}e_z^{(kZ)} \int_{-1}^{1} c(\xi)\eta\sin(\gamma\xi\cos\chi)\, d\xi \quad \text{(a-93)}$$

For the quasi-symmetric vessels to which this paper is restricted this term approaches zero and can be neglected. The heaving force is then

$$F_z = nF = \rho g r_m LBG_z'\cos(\omega_e t - \epsilon) \quad \text{(a-94)}$$

Consider secondly the integral of the horizontal component, which is the swaying force

$$F_y = mF = \rho g r_m \int_{-L/2}^{L/2} \oint m e^{k(Z-h)} \cdot \cos\left[(k\cos\chi)X_s + (k\sin\chi)Y_s - \omega t + \epsilon\right] dl\, dX \quad \text{(a-95)}$$

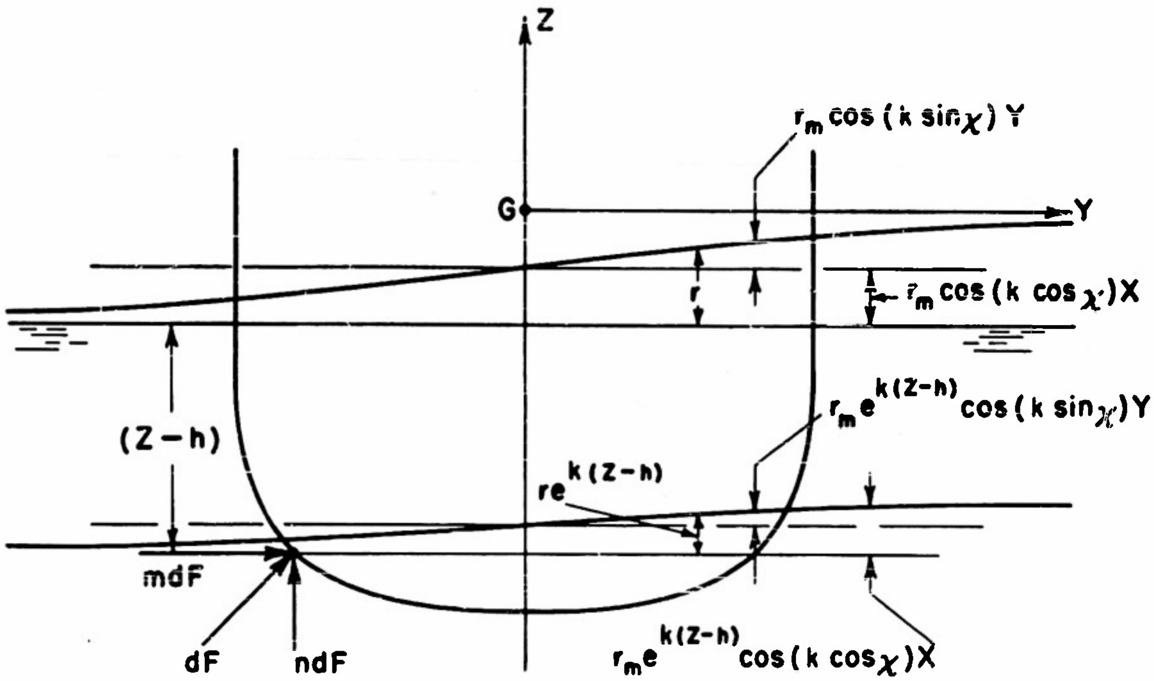

FIG. A-8.—GEOMETRY OF EXCITATION

Expansion of the contour integral gives

$$\sin(\omega_e t - \epsilon) \oint m e^{k(Z-h)} \sin[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl + \cos(\omega_e t - \epsilon) \oint m e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl \tag{a-96}$$

By antisymmetry the first of these terms reduces to

$$2\sin(\omega_e t - \epsilon) \oint m e^{k(Z-h)} \sin(k\cos\chi)X_s \cos(k\sin\chi)Y_s\,dl = 2\sin(\omega_e t - \epsilon)\sin(k\cos\chi)X_s \int e^{k(Z-h)}\cos(k\sin\chi)Y_s\,dZ \tag{a-97}$$

To linearize introduce now the average value of $\int \sin(k\sin\chi)Y_s\,dZ$ at a section. Letting

$$s(X) \equiv s(\xi) \triangleq \frac{\int_0^{Y_s} \sin(k\sin\chi)Y_s\,dY}{\int_0^{Y_s}(k\sin\chi)Y_s\,dY} \tag{a-98}$$

this average value is

$$s(X)\cdot k\sin\chi \int Y_s\,dZ = s(X)A(X)k\sin\chi \tag{a-99}$$

The average value of the exponential being given by equation (a-32), the first term becomes

$2\sin(\omega_e t - \epsilon)e_z^{(kZ)} s(X)A(X)\,k\sin\chi\cos(k\cos\chi)X.$

Integrating along the length and introducing dimensionless parameters, the first term becomes

$LBHe_z^{(kZ)}k\sin\chi \int_{-1}^{1} s(\xi)\beta(\xi)\cos(\gamma\xi\cos\chi)\,d\xi.$

Defining

$$G_y^s \triangleq e_z^{(kZ)}k\sin\chi \int_{-1}^{1} s(\xi)\beta(\xi)\cos(\gamma\xi\cos\chi)\,d\xi \tag{a-100}$$

the first term becomes

$$LBH\cdot G_y^s \sin(\omega_e t - \epsilon) \tag{a-101}$$

Proceeding in a like manner as for the second term of equation (a-96) the complementary antisymmetric term is obtained. This is neglected in view of the restriction to quasi-symmetric ships. The total horizontal force is then

$$F_y = mF = \rho g r_m LBHG_y^s \sin(\omega_e t - \epsilon) \tag{a-102}$$

Consider thirdly the component of the moment to roll arising from the integration of the horizontal component of force.

$$(M_\varphi)_y = \rho g r_m \int_{-L/2}^{L/2} \oint mZe^{k(Z-h)}\cos[(k\cos\chi)X_s + (k\sin\chi)Y_s - \omega t + \epsilon]\,dl\,dX \tag{a-103}$$

Again expanding the contour integral two terms are obtained. They are

$$\cos(\omega_e t - \epsilon) \oint m e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl \text{ and}$$
$$\sin(\omega_e t - \epsilon) \oint m e^{k(Z-h)} \sin[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl. \quad \text{(a-104)}$$

The first leads to an antisymmetric expression which is neglected. The second leads to the complementary symmetric expression

$$LBH \cdot \bar{Z}\, G_y^s \sin(\omega_e t - \epsilon)$$

where $\bar{Z}$ is the ordinate to the center of buoyancy.

The component of the moment to roll from the horizontal forces is then

$$(M_\varphi)_y = -\rho g r_m LBH\bar{Z}\, G_y' \sin(\omega_e t - \epsilon) \quad \text{(a-105)}$$

Consider fourthly that component of the moment to roll arising from the integration of the vertical force components.

$$(M_\varphi)_z = \rho g r_m \int_{-L/2}^{L/2} \oint n Y_s e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl\,dX \quad \text{(a-106)}$$

Expansion of the contour integral gives two terms. They are

$$\cos(\omega_e t - \epsilon) \oint n Y_s e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl \text{ and}$$
$$\sin(\omega_e t - \epsilon) \oint n Y_s e^{k(Z-h)} \sin[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl. \quad \text{(a-107)}$$

The first of these leads to an antisymmetric expression which is again neglected. The second leads to the complementary symmetric expression

$$LB^3 G_\varphi' \sin(\omega_e t - \epsilon)$$

where

$$G_\varphi' \triangleq \frac{1}{24} e_Y^{(kZ)} k \sin\chi \int_{-1}^{1} s(\xi)\eta^3 \cos(\gamma\xi\cos\chi)\,d\xi \quad \text{(a-108)}$$

The resulting moment is

$$(M_\varphi)_z = \rho g r_m LB^3 G_\varphi' \sin(\omega_e t - \epsilon) \quad \text{(a-109)}$$

The rolling moment being the sum of expressions (a-105) and (a-109) is consequently

$$M_\varphi = \rho g r_m LB[B^2 G_\varphi' \sin(\omega_e t - \epsilon) - H\bar{Z}\, G_y' \sin(\omega_e t - \epsilon)] \quad \text{(a-110)}$$

There remains to be considered lastly the moment to pitch. This is due to the vertical force and is

$$M_\psi = \rho g r_m \int_{-L/2}^{L/2} \oint n X e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s - \omega_e t + \epsilon]\,dl\,dX \quad \text{(a-111)}$$

Expansion of the contour integral gives two terms. They are

$$\cos(\omega_e t - \epsilon) \oint n X e^{k(Z-h)} \cos[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl \text{ and}$$
$$\sin(\omega_e t - \epsilon) \oint n X e^{k(Z-h)} \sin[(k\cos\chi)X_s + (k\sin\chi)Y_s]\,dl \quad \text{(a-112)}$$

The first of these leads to a symmetric expression which may be neglected. The second leads to the antisymmetric expression

$$L^2 B G_\psi^a \sin(\omega_e t - \epsilon)$$

where

$$G_\psi^a \triangleq \tfrac{1}{4} e_Y^{(kZ)} \int_{-1}^{1} \xi\eta c(\xi) \sin(\gamma\xi\cos\chi)\,d\xi \quad \text{(a-113)}$$

The pitching moment is then

$$M_\psi = \rho g r_m L^2 B G_\psi^a \sin(\omega_e t - \epsilon) \quad \text{(a-114)}$$

Under certain conditions simplifications in the foregoing formulae for forces and moments can be effected.

(a) When the waves are long with respect to the ship,

$$k \to 0,\ (k\sin\chi)Y_s \to 0,$$
$$s(X) \equiv s(\xi) \to 1,\ c(X) \equiv c(\xi) \to 1 \quad \text{(a-115)}$$

The following expressions are then approached:

Heave.

$$F_z = \rho g r_m LB e_Y^{(kZ)} \cos(\omega_e t - \epsilon) \int_{-1}^{1} \eta \cos(\gamma\xi\cos\chi)\,d\xi \quad \text{(a-116)}$$

Sway—Zero.

Roll—Zero.

Pitch.

$$M_\psi = \tfrac{1}{2}\rho g r_m L^2 B e_Y^{(kZ)} \sin(\omega_e t - \epsilon) \int_0^1 \xi\eta \sin(\gamma\xi\cos\chi)\,d\xi \quad \text{(a-117)}$$

These expressions for heave and pitch are in agreement with equations (96) and (175) of reference [20].

(b) When long seas approach from the beam, $\cos\chi \to 0$, $\sin\chi \to 1$, $\cos(\gamma\xi\cos\chi) \to 1$, $s(X) \equiv (\xi) \to 1$, $k\sin\chi \to k$, the following expressions then result.

Heave.

$$F_s = \rho g r_m A(Z) e_Y^{(kZ)} \cos(\omega_e t - \epsilon) \quad \text{(a-118)}$$

Sway.

$$F_y = \rho g r_m k \nabla e_Z^{(kZ)} \sin(\omega_e t - \epsilon) \quad \text{(a-119)}$$

Pitch—Zero.
Roll.

$$M_\varphi = \rho g r_m k \,[J_z e_Y^{(kZ)} - \nabla \bar{Z} e_Z^{(kZ)}] \sin(\omega_e t - \epsilon) \quad \text{(a-120)}$$

This expression is similar to that for the righting moment, equation (a-77). Writing the angle of roll

$$\varphi = \varphi_m \sin(\omega_e t - \epsilon)$$

and assuming that $e_Y^{(kZ)} \cong e_Z^{(kZ)} \triangleq e^{(kZ)}$ the equivalent terms are

$$\varphi \sim r_m k e^{(kZ)} \quad \text{(a-121)}$$

Since $[J_z - \nabla \bar{Z}] = \overline{GM} \cdot \nabla$ the above expression for roll may be written

$$M_\varphi = \rho g \nabla \cdot \overline{GM} \cdot r_m k e^{(kZ)} \sin(\omega_e t - \epsilon) \quad \text{(a-122)}$$

which corresponds to equation (96) of Reference [24].

SOLUTION

The hydrodynamic coefficients that have been derived are now entered into the equations of motion (a-1). If these are divided through by the virtual mass or moment of inertia, they take the form

$$\frac{d^2 s}{dt^2} + 2h_s \frac{ds}{dt} + \nu_s^2 = f_s \binom{\sin}{\cos} (\omega_e t - \epsilon) \quad \text{(a-123)}$$

where

$$2h_s \triangleq \frac{N_s}{M_s, I_\psi, I_\varphi} \quad \text{(a-124)}$$

as the case may be

$$\nu_s^2 \triangleq \frac{R_s}{M_s, I_\psi, I_\varphi} \quad \text{(a-125)}$$

and

$$f_s = \frac{F_s}{M_s} \text{ or } \frac{M_\psi}{I_\psi} \text{ or } \frac{M_\varphi}{I_\varphi} \quad \text{(a-126)}$$

For longitudinally symmetric vessels the solution to equation (a-123) is

$$= \frac{f_s}{\nu_s^2} \cdot \mu_s \cdot \binom{\sin}{\cos} (\omega_e t - \epsilon - \epsilon_s) \quad \text{(a-127)}$$

where μ_s is a magnification factor given by

$$\mu_s \triangleq \frac{\nu_s^2}{\sqrt{(\nu_s^2 - \omega_e^2)^2 + (2h_s\omega_e)^2}} = \frac{1}{\sqrt{(1 - \Lambda_s^2)^2 + \kappa_s^2 \Lambda_s^2}} \quad \text{(a-128)}$$

with

$$\Lambda_s \triangleq \frac{\omega_e}{\nu_s} \quad \text{(a-129)}$$

and

$$\kappa_s \triangleq \frac{2h_s}{\nu_s} \quad \text{(a-130)}$$

where f_s/ν_s^2 is a force factor for unit wave amplitude, such that

$$\frac{f_s}{\nu_s^2} \triangleq r_m E_s \quad \text{(a-131)}$$

and where ϵ_s is a phase lag given by

$$\epsilon_s = \text{arctg}\, \frac{\chi_s \Lambda_s}{1 - \Lambda_s^2} \quad \text{(a-132)}$$

The general solution can therefore be written as

$$s = r_m E_s \mu_s \binom{\sin}{\cos} (\omega_e t - \epsilon - \epsilon_s) \quad \text{(a-133)}$$

or, specifically

$$\left.\begin{aligned} z &= r_m E_s \mu_s \cos(\omega_e t - \epsilon - \epsilon_s) \\ \psi &= r_m E_\psi \mu_\psi \sin(\omega_e t - \epsilon - \epsilon_\psi) \\ \varphi &= r_m E_\varphi \mu_\varphi \sin(\omega_e t - \epsilon - \epsilon_\varphi) \end{aligned}\right\} \quad \text{(a-134)}$$

where

$$E_s = \rho g L B C_s' \quad \text{(a-135)}$$
$$E_\psi = \rho g L^2 B G_\psi^a \quad \text{(a-136)}$$
$$E_\varphi = \rho g L B [B^2 G_\varphi' - B\bar{Z} G_z'] \quad \text{(a-137)}$$

DISCUSSION

MR. B. V. KORVIN-KROUKOVSKY, *Member:* This excellent paper consists of three major parts: The first gives the mathematical description of a complex sea surface by summation of an infinite number of simple harmonic waves; the second deals with the response of a ship to a simple harmonic wave; and the third gives the complex ship motion by summation of motions due to simple waves. The first two parts are entirely independent of each other, and therefore require

separate statements as to the assumptions made and as to the degree of the validity obtained.

The only basic assumption made in the first part is that a complex form of sea surface can be obtained by linear superposition of simple waves. This appears to be a valid assumption involving but a negligible error, since the final wave within the range of practical interest does not differ too much from a trochoidal one of 1:20 height to length ratio, the maximum surface slope of which is only about 9 deg. For a sea generated by a steady wind the points from a wave record are said to conform to the normal probability curve, and this can be reasonably accepted as a universal law of nature. This cannot be said about the distribution of the angles χ of the directions of wave propagation which are responsible for the short-crestedness. While the authors stress the short-crestedness, they give practically no information on the choice of the range of angles χ to be expected in practice. Yet the most complicated section of the entire paper, that on the frequency mapping, seems to deal with the significance of these angles. Hence a more thorough discussion of the origin and magnitude of these angles appears to be desirable.

The second part of the paper, dealing with ship response to a simple harmonic wave, is said to have two basic assumptions; that the different modes of the motion are not coupled, and that they can be described by linear differential equations. These statements cannot be taken to apply to the entire second part of the paper, but rather to the mathematical calculation of ship response, by which the authors seem to be fascinated to the detriment of the fair evaluation of experimental methods.

To start with it is said that only three modes of motion will be considered, that of pitch, heave, and roll. Theoretical work of Haskind[13] indicates that pitch and heave are uncoupled only for a ship not moving, but are coupled at a forward speed even for a ship symmetrical fore-and-aft. However, the degree of error in considering them uncoupled is not known, and the authors have a good deal of past tradition in considering them as uncoupled. This cannot be said, however, about the roll, which is strongly coupled with the yaw, as this was forcibly brought out in the case of the *Conte di Savoia*. Furthermore, the inertial characteristics of ships provide a coupling of pitch, roll, and yaw in quartering sea. This being the case, theoretical methods of this part of the paper may be accepted for the ship motions in pitch and heave only in case of the head or following sea, but cannot be considered as valid either for the roll or for the pitch and heave in oblique sea.

In the discussion of experimental methods the authors skip briefly over the evaluation of ship response by direct test and devote considerable space to the experimental determination of various coefficients needed for use in the linear differential equations of motion. The writer appreciates the quality and completeness of this exposition, but feels that it is irrelevant to the subject of the present paper, since the response can be evaluated by the direct test with a lesser number of limiting assumptions.

A series of tests can be made by towing a model with or against regular waves of different lengths. If the available towing distance is sufficient the model likewise will have a simple motion. The ratio of the maximum model heave to the wave height, of maximum pitch to wave slope and corresponding phase lags can be measured readily. The plots of these quantities versus frequencies of wave encounter will give directly the ship-response functions needed for the application of the authors' theory. It should be noted that while the data so obtained are given separately for heave and pitch, they are taken from the records of the actual coupled motion, and therefore do not contain the limiting assumption of noncoupling. By the same token, when the towing-tank equipment will permit a model path oblique to the direction of wave propagation, the effects of the pitch-roll-yaw coupling will be represented.

In the foregoing, the obvious method of measuring experimentally ship response at varying frequencies was described, as this method is immediately available with the use of the existing towing-tank equipment. Attention should be called, however, to the paper by Fuchs and MacCamy[14] in which it is shown that response functions can be obtained from the analysis of ship-model motions in a complex sea. One still has to wait, however, for a satisfactory method of generating a realistic complex sea in a towing tank.

In the third part of the paper, occurs the assumption of linearity necessary for obtaining the complex ship motions by summation of elementary motions. This is the most serious limiting assumption of the paper. The mathematicle evaluation of the ship response is necessarily limited to it. Yet, model tests in 1:20 wavsy show the pitching to be so violent that alternate-e the water is taken over the bow and an appral

[13] "Two Papers on the Hydrodynamic Theory of Heaving and Pitching Ship," by H. D. Haskind, SNAME Technical and Research Bulletin No. 1-12.

[14] "A Linear Theory of Ship Motion in Irregular Waves," by R. A. Fuchs and R. C. MacCamy, University of California, Institute of Engineering Research, Technical Report Series 61, Issue 2, July, 1953.

ciable length of the keel is exposed. Clearly there can be no linearity here. In case the direct experimental method is used, the situation is ameliorated by using the mean value of the response determined from the motion of the violence found in practice.

The direct experimental determination of ship-response function has been stressed in the foregoing because the methods of measurement and the equipment needed are immediately available, and because it increases the power and validity of the authors' theory by taking into account the cross-coupling effects of various modes of motion, and by taking care partially of nonlinearity by using mean values of the response for the motion of a practical amplitude. However, a further development of theoretical means of calculating ship response is evidently needed in order to show not only what is happening, but also why it is happening. The problem of ship motions, as distinct from the sea conditions, has to be recognized, however, as a definitely nonlinear one.

Two directions of action appear to be possible. Mathematicians probably can tell us to what extent the solution of a violently nonlinear problem can be approximated by using the mean values of functions for the estimated amplitude of motion. This actually is done when the experimental method is used, but in this case the amplitude of motion is determined and need not be estimated. If this approach can be justified on theoretical grounds, the analytical calculations of ship response likewise can be directed to computation (probably by numerical integration) of mean values.

The alternate and more precise approach is a step-by-step integration of nonlinear equations of motion, as for instance was done by Hazen and Nims.[15] It has been assumed by Kriloff[16] and recently restated by Weinblum and St. Denis[17] that the most important part of the forcing function in a seaway is due to the changes of hydrostatic pressure gradient in waves as compared to the one existing in still water. These changes are well known for a simple trochoidal wave, but are not directly available for a complex sea.

It appears that a valuable extension of the authors' method of describing a complex sea surface would be to give the expressions for a corresponding pressure gradient at depth. The nonlinear forcing functions and the ship motions can then be obtained by a numerical step-by-step integration, probably using electronic computing machines. This would lead to the specific description of the ship motion, including the determination of the phase lag, which is extremely important in determination of accelerations, bending stresses, and wetness.

The present paper is believed to be the first attack on the problem of ship motions made jointly by a naval architect and an oceanographer, and the authors are to be congratulated on the excellent results they have achieved. It is hoped that they will continue the development of this subject.

Mr. William E. Cummins, *Member:* This paper is a pioneering investigation which introduces a new concept into the study of seaworthiness. The idea of describing the motion of a ship in a seaway in terms of its energy spectrum is a radical departure from previous methods employed in this field, and is potentially extremely useful in many ways. The importance of this procedure derives from the fact that if the energy spectrum of a random process is known, the statistics of the process are essentially determined. Since the concept of the energy spectrum is certainly the best mathematical tool at present available for describing random phenomena of this sort, the paper is of fundamental importance, and will be valuable for all future research.

It may be pointed out that the implications of this concept extend well beyond the particular problem considered here, since any oscillatory function A, which is a function of a stationary random process B, can be described in terms of an energy spectrum which in general is a unique function of the energy spectrum of B. It may be difficult to determine this functional relationship, but the existence of the relationship is itself significant. Thus, if the nonlinearities of the equations of motion are such as to invalidate the authors' analysis, as might well be the case under storm conditions, the solution still exists, and it may be found if the investigator is sufficiently clever. Thus, the neglect of nonlinearities, coupling, and other phenomena is not a limitation of the concept of the energy spectrum, but only of the particular study being undertaken.

It is regrettable that the authors have not indicated possible applications of this rather difficult theory, since in that way they could have alleviated the pain of following through the analysis. Actually, there are numerous possible applications which might well be of interest both to the naval architect and the ship operator.

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[15] "Calculation of Motion and Stresses of a Pitching and Heaving Ship," by H. L. Hazen and P. T. Nims, SNAME *Transactions*, vol. 48, 1940, pp. 94–113.

[16] "A New Theory of the Pitching Motion of Ships on Waves and of the Stresses Produced by This Motion," by A. Kriloff, INA, vol. 37, 1896, pp. 326–359.

"A General Theory of the Oscillations of a Ship on Waves," and "On Stresses Experienced by a Ship in a Seaway," by A. Kriloff, INA, vol. 40, 1898.

[17] Authors' bibliography, refs. [20] and [24].

For instance, the naval architect has always been handicapped by the lack of criteria for evaluating the seaworthiness characteristics of proposed designs. He has fallen back upon experimental studies of models in regular waves (either from ahead or astern) simply because there was no alternative for getting quantitative data, although he was fully aware of the limitations of such data. Now, we can look forward to something better.

When the oceanographers can present us with reasonably good mathematical descriptions of certain typical sea states, and when adequate analytical or experimental facilities are available for predicting the motion of a vessel in a regular sea from any direction, the naval architect can use a much more rational approach. He can pick out several more or less realistic mathematical models of typical seaways in which the vessel is expected to operate, (say winter North Atlantic storms of various degrees of intensity) and determine the effect of loading, course, form, and so on, upon the behavior of the proposed vessel under these conditions. Such a process need not be very accurate in order to be superior to present methods, so the mathematical models need not be too precise. The designer would then have meaningful criteria in the form of average period, amplitude, standard deviation, and so on, for predicting the ship's behavior under conditions which are likely to occur in service.

If the ship operator is considering an attempt to modify the behavior of a vessel by adding ballast (a very expensive expedient), he is likely to be interested in a comparison between the results of such a procedure and those which can be obtained simply by altering the course slightly. The methods proposed in this paper should provide such information.

Another possible application is the correlation of stresses in structural members with the sea state, since such stresses also will have their energy spectra. If these spectra could be determined from the spectrum of the seaway, they might be useful in studying both fatigue and probable maximum stress.

The authors have gone a long way towards solving theproblem of finding the energy spectrum of the ship's motion. In doing this, they have made certain assumptions, which in general, are reasonable, and at the present time probably necessary. However, since these assumptions provide the principal limitations of the analysis, I should like to make some remarks about them.

The authors have restricted their development to uncoupled motions, with the statement that this limits the theory to vessels whose waterlines are quasisymmetric fore and aft. However, Haskind has shown that heaving and pitching oscillations are coupled, even in vessels which are symmetrical fore and aft.[18] It would be interesting to see the effect of coupling upon the predicted energy spectrum.

The authors make use of the Froude-Krylov hypothesis to determine the exciting force. As they indicate, no attempt has yet been made to evaluate this hypothesis. Since at the David Taylor Model Basin we are carrying out certain studies which are related to this problem, some comments about the Froude-Krylov theory based on the results of these studies might be of interest.

The exciting force is due to two separate effects acting simultaneously: (*a*) There is a shift in the distribution of the buoyant forces, since the wave profile results in a change in the shape of the submerged volume; and (*b*) the pressure distribution over the hull is modified by dynamic effects due to the motion of the water. In the Froude-Krylov theory, the wave profile is assumed to be due to the seaway alone, unaltered by waves generated by the vessel, and the pressure acting at any point is considered to have the same value it would have if the vessel were not present. Hence, the statement, "the waves act on the ship but the ship does not act on the waves."

We have investigated the forces acting on submerged bodies of revolution moving with a constant horizontal velocity under a train of regular waves. The changes in the flow due to the presence of the body have been taken into consideration, but the waves created by the body have not. Since there are no complications because of a changing wave profile, we have been able to draw certain conclusions about effect (*b*). In general, the vertical force involves a primary term which is proportional to the wave amplitude and a secondary term which is proportional to the square of the wave amplitude. The primary term is a linear function of the ratio of the velocity of the body to the celerity of the waves. Surprisingly, the effect of a change in this ratio is much greater for slender bodies than for short, blunt bodies. For long, slender bodies, this term is zero when the body velocity is twice the wave velocity. The secondary term, which is independent of the forward velocity, is usually small, but for high waves it can assume significant values.

From these results, we can conclude that the Froude-Krylov theory should be used with some caution. Since this theory predicts forces which are independent of the speed of advance, its treatment of the dynamic effects is completely

[18] Technical and Research Bulletin No. 1-12, The Society of Naval Architects and Marine Engineers.

inadequate, and in cases where these are important, it might lead to serious errors. To the extent that the exciting force is due to the change in the distribution of buoyancy, which may be the dominant effect for surface vessels, it is probably much more reasonable.

The occurrence of terms involving the square of the wave amplitude is also of significance. While the authors have limited their discussion to *small* waves and *small* motions, I trust that they will not so limit themselves in their applications of the theory, since it is the large motions which are of interest both to the naval architect and the passenger. Therefore, nonlinear effects predicted from the linearized theory may deserve some attention. The existence of these terms also indicates that when two wave trains of different periods are superimposed, the resulting exciting force will include certain "coupling" terms as a result of the interference of the two systems. Thus, there may be an appreciable error merely in adding the separate responses to find the combined response.

MR. EDWARD V. LEWIS, *Member:* This excellent paper is the outcome of a most successful collaboration between an oceanographer and a naval architect. When all the implications have been explored, it undoubtedly will prove to be of far-reaching significance in naval architecture, particularly it is believed from the point of view of making possible improvements in hull design which will in time result in important advances in the all-weather speeds of ships of all types. Log data show that in really bad weather services speed loss is not due mainly to added resistance but to voluntary reduction of power to reduce the violence of the ship's motions. Consequently, our interest in the motions of ships is not simply in making ships more comfortable, but in reducing or modifying the motions in a seaway so that full power can be maintained in the face of more severe sea conditions.

From the point of view of the practical naval architect then, it would seem that the most significant contribution of the paper is in the thesis that the response of a ship to a confused sea need not be dealt with as a continuous succession of transients, which would be hopeless to analyze, but as a steady-state process; that the response is equal to the sum of the separate responses to a large number of simple waves.

This opens the way for the immediate advance of our knowledge of ship motions. Hitherto, we have been uncertain as to how to interpret the results of model tests in regular waves, but now the possibility is provided of using the model tests in regular waves as the building blocks (response-amplitude operators) to construct the expected behavior of the ship in statistical terms in any except extreme complex sea situations. Of course, the foregoing hypothesis must be verified in some way. One method would be to use model tests in complex seas, and it is to be hoped that facilities for tests under such conditions will soon become available somewhere. The question will then be as to how "complex" the tank waves must be; e.g., how many components are needed to create a sea which *for the purpose of ship motions* is satisfactory.

The criterion is clearly not the same as the oceanographer's criterion in attempting a scientific representation of the surface down to the last ripple. Even if the theory is found to apply only to comparatively low seas, it undoubtedly will be of great value in obtaining qualitative indications of the effects of modifications of hull form and proportions on ship motions at sea.

Let us consider now the basic problem of determining the "building blocks" or response-amplitude operators, which may be visualized as a family of curves showing the amplitude of pitch, for example, in waves of say 1 ft height (full scale) and varying length, with a different curve for each of several ship speeds and for various headings. The best available method for determining such curves is by model tests, since they involve no artificial assumptions and give the complete picture regarding the interaction or coupling between pitch and heave. It is not meant to minimize the importance of the development of theoretical methods of calculation of the response-amplitude operators, the problem to which one of the authors has devoted so much effort. However, until they have been confirmed by model tests, as contemplated in Mr. St. Denis' earlier paper,[19] the writer does not see how we can adopt them in preference to the direct experimental determination. The methods of the paper appear much stronger if not made to depend upon the accuracy of theoretically calculated operators.

Incidentally, the paper does not seem to do the experimental methods full justice, since the "severely restricted conditions" for pitch and heave are certainly the most important conditions to be studied--head and following seas. Facilities for model testing in regular oblique seas can and should be provided to furnish the operators for other directions of motion of the ship relative to the waves. Meanwhile, however, approximations for pitch and heave are obtainable in a conventional straight tank in the following

[19] Authors' bibliography, ref. [20].

way, at least for heading angles within about ±45 deg from bow or stern. For a heading angle χ, the model is tested head-on into substitute waves of length, $\lambda/\cos\chi$, and the difference in the wave profiles on the port and starboard sides due to obliquity of heading is neglected.[20] The period of encounter will be different from that desired, since the velocity of the substitute wave will be higher than one of length λ. Hence, the model speed must be modified to give the correct period of encounter. (For a sea 45 deg off the bow this amounts to a 50 per cent increase in speed.) How serious the error may be in using the incorrect speed is of course uncertain.

Another method of obtaining the operators in a straight tank can be used if the part of the theory can be verified which says that the operators can be separated into a magnification factor μ_s, depending only on frequency of encounter, and a factor E_s, which is a function of effective wave length and ship speed equation (2.4) of the paper. If the damping coefficient also can be obtained by a separate test in the manner described in the paper, it is a simple matter to compute the μ_s as tested and as desired to correspond with the correct period of encounter. The operator A_s determined at correct speed can then be corrected by multiplying by the ratio of the μ_s-values. However, since the corrections will be different for pitch and heave, the existence of coupling between the motions will introduce complications which may be serious. At any rate, we do not need to wait for either wide tanks or the perfection of the analytical calculation of response-amplitude operators to apply the powerful methods presented in this paper.

Coming now to the mathematical representation of the sea surface presented in the paper, it seems to have most of the properties of the actual sea, and is undoubtedly much more realistic than other schemes. However, the writer is sure the authors do not mean to give the impression that the problem of ocean wave patterns has been solved completely—as the opening quotation says, "Mathematics can never tell you what is; only what would be if." It is hard to compare presently available data on ocean waves with this theory, expect perhaps wave heights. Here we find that the theory gives much lower heights at moderate wind velocities than Kempf's chart for the North Atlantic. For a 20-knot wind, reference [15] of the paper gives an average wave height of 5 ft and a "significant" wave height of 8 ft for a fully developed sea, whereas Kempf gives a height of 14 ft. With increasing wind velocity the results come closer, so that for a 40-knot wind an average height of 28 ft and significant height of 45 ft agrees reasonably well with Kempf's 40 ft. On the other hand, the average heights agree quite well with Kent's formula, wave height in ft = K (true wind speed in knots)^2, where $K = 1/50$ and $1/64$ for winds below and above 20 knots, respectively. Thus, the paper provides additional evidence that average wave heights in moderate winds are lower than generally has been believed.

For other reasons than wave height, the theoretical spectrum for a short-crested sea (Fig. 1.6) must be used with caution. It represents an idealized case; a steady wind blowing for a long time over an initially calm surface. For ship-design problems we need to know a great deal more about the typical values of fetch and duration to be found on the various sea routes of the world, since these determine the maximum development of the spectrum to be expected. Furthermore, on stormy routes such as the North Atlantic, storms rarely if ever build up a sea over a calm surface. We would expect a swell to be present from a previous storm, for instance, and the combined spectrum certainly would be much different from the ideal.

Kent describes the typical storm condition in the North Atlantic as one of constantly changing wind force and direction. When the wind direction shifts suddenly, a second—and sometimes a third—cross sea soon builds up forming "hummocks" much greater in height than normal for that wind. It might well be that the condition could be described adequately much more simply by two long-crested seas of different direction. It seems then that observational data on the actual spectra observed on different routes are needed, the theory of the first section of the paper providing an invaluable foundation for detailed sea observation. It is to be hoped that the oceanographers can fill this gap for us.

Realizing that ship responses to simple waves may be secured readily from model tests and that typical ocean-wave spectra are obtainable, we are still faced with the somewhat involved problem of the frequency mapping, if we are to obtain the ship response to a confused sea. This complication comes about as a result of the authors' rigorous treatment of short-crested seas, involving an angular integration with varying frequencies of encounter. The question arises as to the significance of the short-crestedness for the problem of ship motions, particularly the motions of pitch and heave. As a practical matter, most of the wave energy in Fig. 1.6 seems to be concentrated within ±30 deg of the dominant direction. Considering the situation carefully, one may conclude

[20] Authors' bibliography, ref. [24], p. 204.

that for head and following seas, the difference between using a short-crested sea or one with infinitely long crests would be insignificant, since the maximum variation in effective wave lengths and periods would be by a factor of cos 30 deg or 0.86. For other wind directions the cosine error increases; with a beam sea, the long-crested case would give a zero pitching moment, for example. For other than head and following seas, the writer suggests that two or three long-crested seas of different direction might be selected which would be essentially equivalent to the ideal short-crested sea and simplify the problem greatly. Then a simple addition of terms would replace a complicated mapping and angular integration, and even a cross sea would not introduce serious complications. Ships seem to be so sensitive to resonant periods (as discussed later) that fairly large changes in the wave spectrum may not have appreciable effect on the response spectrum. At any rate, consideration should be given to other important effects—as the cross seas previously mentioned—and some actual cases should be worked out numerically to determine the significance of the rigorous short-crested representation.

Whether or not long-crested seas can be used in all cases, there can be no doubt that dealing with the simplified case of a single long-crested sea is most helpful for gaining an understanding of the principles involved. To this end it would be useful to have a scale added to Fig. 1.8.

Using numerical values from reference [15], a few examples have been worked out for a typical ship using approximate response-amplitude operators from a model test, which show clearly that in a storm with winds of 30 knots or more—if the spectra of Fig. 1.8 are typical—the response spectra are much narrower than the wave spectrum, with the peak close to the synchronous frequency. This serves to explain the often reported fact that in heavy seas ships usually roll and pitch very close to their natural periods. The methods of this paper supply the means of studying this problem in detail.

If it is established that storm spectra are really broad and continuous, i.e., that there is always a wide range of wave frequencies present, this fact changes our whole viewpoint on ship motions. We then no longer can hope to find dominant wave lengths in a confused sea or expect a design based on some average wave length and height to be the optimum. Furthermore, it so happens that the natural pitching and heaving periods of ships are such that at normal speeds they are likely to coincide with the periods of encounter of some components of a typical storm sea as described by the authors' spectra, no matter what the heading. Hence, the difficulty for a normal ship to avoid serious pitching and heaving in a real storm is clearly explained.

For example, the 30 knot spectrum of Fig. 1.8, shows that the important frequencies present correspond to wave lengths ranging from 200 to 1500 ft, of which those from 300 to 1200 ft would be expected to exert large pitching moments on a 500-ft ship.

Valuable as this paper is, there are one or two things the naval architect still needs in order to solve the problems of improved sea-keeping qualities. One is a knowledge of the accelerations as well as the amplitudes of motion. Another is to know the relative position of the ship in relation to the wave pattern producing the motion. The latter is needed for judging the wetness of decks and the tendency of the propellers to race; it is needed for calculating stresses in complex seas and investigating the phenomenon of slamming. Although the paper suggests the possibility of more complicated statistical methods of getting at the phase relationships, a simpler technique of determining the instant-by-instant motion of the ship in relation to a typical complex wave pattern of any desired length would suffice.

It may be of interest to add a brief summary of the simplified method of calculation resulting when a single long-crested random sea is used, and perhaps the authors can correct any oversights. The seaway is represented by equation (1.18) of the paper, for a definite value of heading angle χ. The wave spectrum at a fixed point $[r(\omega)]^2$ is of the form of Fig. 1.8. When transformed to the axes moving with the ship, equation (1.18) takes the following form, derived from equation 3.12 with X_e and Y_e put equal to 0 (since we are dealing with the motion of the center of mass of the ship);

$$r(t) = \int_0^\infty \cos\left[-\omega_e t + \epsilon\right] \sqrt{[r(\omega_e)]^2 \, d\omega_e}$$

in which ω_e is given by equation (3.2). The spectrum $[r(\omega_e)]^2$ is obtained from $[r(\omega)]^2$ very simply by replotting points on the spectrum curve for different values of ω at the correct value of ω_e equation (3.2), and multiplying by the Jacobian. The value of α_e is obtained from equation (3.15) for each value of ω_e. (For head seas the spectrum

is thereby widened; for following seas narrowed.) In the case of following seas, the necessity of dividing the spectrum into regions I, II, and III becomes immediately evident, for the curve usually doubles back on itself and may extend into negative frequencies. (In the case of head seas, only region I is involved.)

Points on the ship-response spectrum are then obtained, from $[s(\omega_e)]^2 = [r(\omega_e)]^2[(A_s)(\omega_e)]^2$ (eq. 4.10 simplified) (with a minus sign in region III), since the definition of $[r(\omega_e)]^2$ already involves the Jacobean. Here $[(A_s)_e(\omega_e)]^2$ is the squared value of the response-amplitude operator, obtained by calculation, model test, or a combined method—for a particular value of wave length, ship speed, and heading angle (hence fixed χ_e). The operator will be in the form of degrees of pitch, or feet of heave, per foot of wave height.

The quantity R^* is finally obtained by evaluating equation (4.14)

$$R_s^* = \int_0^\infty [s(\omega_e)]^2 \, d\omega_e$$

If other than region I is involved, each must be dealt with separately, as described in the paper. The average amplitude, "significant" amplitude, and so on may then be obtained from R_s^* by the methods of the last section (equations 5.11, 5.12, 5.13).

C. O'D. Iselin,[21] *Visitor:* Some rather novel measurements are now being made in order to furnish suitable data for testing this most promising new theory. The work is being carried out under a 2-year contract between the Woods Hole Oceanographic Institution and the Office of Naval Research, and has been in progress for about 6 months.

A 36-ft launch has been selected and equipped with means of recording pitch roll and heave. Two methods of recording the statistical characteristics of the waves in which the launch is operated have been developed. A capacitance-type wave recorder will give the energy spectrum of the waves at a single point. Since the observations are being made under a high bridge across Narragansett Bay, by means of stereophotography it is also possible to observe the two-dimensional characteristics of the waves in which the boat is operating. The response-amplitude operators of the launch are calculated or measured directly.

The variables then are the characteristics of the sea surface in the operating area, the speed of the launch and the angle of attack. Trim and ballasting remain fixed. Provided the accuracy of each of the several types of observations secured simultaneously can be demonstrated clearly, the differences between the observed motions of the launch and the motions predicted by the authors' theory will depend on the hull-induced modifications of the natural wave systems present. This means that agreement between the measurements and the theory should be best at low speeds and in relatively small waves.

Some of the practical difficulties that have been encountered will perhaps be of interest. It has been found necessary to work at slack water for several reasons. Since the wave pole is moored, only at slack water is the record free of Doppler effect for the smaller waves present. But perhaps of more importance is the effect of the tidal currents on the course of the launch when running at an angle across the bay. If her heading relative to the wind is kept constant, as it must be, it is most difficult for the boatman to judge the set of the current so as to pass under the center of the bridge.

Each run lasts 7½ min in order to provide an adequate record of pitch, roll, and heave. Since each set of observations involves runs on five headings, if no mistakes are made, it takes at least 40 min to secure a complete series of data. Furthermore, during this time the characteristics of the waves must remain constant. This means that the wind must continue in the statistical sense steady in direction and strength, and that there be no boat traffic in the area. During the summer months it was not practical to await times when all these conditions were fulfilled in all respects simultaneously.

The instrumentation has been checked thoroughly and one complete set of observations at 6 knots in 3-ft waves has been obtained.

One principal uncertainty remains. We do not know for certain under what circumstances the significant statistical characteristics of small waves approach those of a fully developed sea. After a good deal of effort it now appears that we can obtain reliable measurements of the waves passing under the Jamestown bridge, but no such high-quality data are yet available from the open ocean. This perhaps is not critical for the primary purpose of the present measurements, for the theory should give the response of the launch

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[21] Woods Hole Oceanographic Institution, Woods Hole, Mass.

in question to any sorts of waves, but of course it also would be desirable to test the theory under conditions approximating storm waves.

Under a contract with the Bureau of Ships work also is in progress at Woods Hole on the development of a reliable open ocean wave recorder. Enough experience already has been gained to show that it will not be easy to develop an instrument capable of recording extreme conditions. This is especially the case, if the over-all dimensions of the instrument are to be kept to reasonable limits. However, if the seakeeping problem is to be advanced, it is clearly desirable to obtain as rapidly as possible some complete measurements of the characteristics of storm waves.

An oceanographer is accustomed to having to secure his measurements when and if Nature offers a favorable and conclusive opportunity. Nevertheless, it must be admitted that the approach which has been outlined, involving measurements of the motions of a small, practical vessel in relatively large natural waves, requires considerable patience. However, in theory at least it overcomes some of the fundamental difficulties of studying ship motion in the towing tank.

Dr. K. S. M. Davidson, *Vice-President:* This paper is tangible evidence of what I am convinced has long been overdue; namely, a close collaboration between an oceanographer and a naval architect, working together on the subject of ship motions. I am naturally pleased to know that some remarks of mine may have been partly responsible, as the authors suggest.

I believe that this collaboration represent a milestone in the development of a proper understanding of the seakeeping characteristics of ships. Its full implications probably will not be evident for some time. The important thing now is that a start has been made. If this start looks unduly complicated, I would say only this: that surely most of the past work on the seakeeping characteristics of ships has been unduly oversimplified.

I will confine myself to one or two brief general comments on the paper.

As I understand it, there are three basic concepts, which, taken together, form an orderly and complete pattern:

1 The mathematical representation of the confused sea surface.

2 The analytical determination of the responses of the ship or model to simple waves.

3 The statistical determination of the responses of the ship to confused seas.

Considering first the representation of the confused pattern of the sea surface, the method presented appears to be important in providing a coherent mathematical framework on which to build. What is needed now seems to be more information from the oceanographers regarding the actual sea spectra likely to be met with under various conditions. I find it very difficult to believe, however, that completely random seas are needed to meet the requirements of realistic studies of ship motions, and it is obvious that reduction to a (small) finite number of wave components would effect a very considerable simplification for the naval architect attempting to improve the seakeeping characteristics of his ships.

Turning next to the determination of the responses of the ship or model to simple waves, this is an area in which the model test is already established. I suspect that, assuming the existence of facilities for conducting tests at angles other than 90 deg with the wave crests, the model test will remain for some time to come the principal source of the needed information. However, it is unnecessary to argue here the relative advantages of analysis and experiment for the purpose. The important point of the paper is the concept that the response to *simple* waves is all that is needed.

Coming now to the third concept, that of determining the responses of the ship to confused seas by the summation of the responses to simple waves, this strikes me as perhaps the most significant concept of the three. If it can be established as valid, then with the first two concepts, a complete pattern is at hand for assessing the relative motions of different ships at sea and of evaluating their relative seagoing characteristics.

The thesis as a whole rests on the basic notion that ships at sea may behave very differently from models in simple (regular) waves. I subscribe fully to this notion, and have said so before now. But whether the pattern that the authors have proposed for getting out of the difficulty is workable, or will in itself lead to the desired end, remains to be seen. I suspect that it is only one of several approaches that will need to be explored thoroughly before we know where we are with assurance.

On one point, however, there can be little question. Far-reaching analyses of some sort are surely needed if we are to advance our understanding of an obviously difficult subject, and put ourselves in a position to take advantage of the advances in power-plant design that are certain to come in the near future. We cannot proceed by empirical tests alone.

I trust that the authors will accept my congratulations on the work they have done. And I echo their hope that collaboration between oceanographers and naval architects will continue, to the pleasure and profit of both.

MR. V. G. SZEBEHELY, *Member:* The authors are to be congratulated for attempting a paper on an interesting and difficult subject; a paper worthy of, as well as requiring, much study.

This discussion takes up three questions, each of immediate interest to the naval architect. The first comment is of a theoretical nature. It questions the necessity of the use of the energy integral and suggests a possible substitute which is somewhat less sophisticated and more commonly known to the engineer. The second comment deals with the Gaussian property of the confused sea and describes some recent attempts at the David Taylor Model Basin to produce such sea conditions. The third comment deals with the application of the statistical approach to an actual problem in seaworthiness.

1 The following oversimplified discussion intends to show how the response of a linear system to a random input can be computed without making use of the energy integral. The simplification consists of concentrating on the differential equation of the motion of the ship and neglecting the details of the response-amplitude operator and the frequency mapping. The basic idea of the following method is not changed if these effects are included, but the essential features might become less clear.

Using the author's notation we find the steady-state solution of the linear differential equation of the motion of a ship, first assuming a pure cosine input

$$\frac{d^2s}{dt^2} + 2h_s\frac{ds}{dt} + \nu_s^2 s = f_s \cos \omega t = F_s(t) \quad (1)$$

$$s(t) = B_s \cos (\omega t + \epsilon_s) \quad (2)$$

Equation (2) is a solution of equation (1). Here s represents the response (heave, pitch, or roll), f_s the input amplitude and ϵ_s the phase lag. The output and input amplitudes are connected by the elementary formula

$$B_s^2 = \frac{f_s^2}{(\nu_s^2 - \omega^2)^2 + (2h_s\omega)^2} = \frac{f_s^2}{|z(\omega)|^2} \quad (3)$$

where $|z(\omega)|^2$ is the absolute square of the impedance of the system.

The mean square of the input is

$$\overline{F_s^2} = \frac{1}{2T}\int_{-T}^{T} (f_s \cos \omega t)^2\, dt = \frac{f_s^2}{2}$$

and that of the output

$$\overline{s^2} = \frac{1}{2T}\int_{-T}^{T} [B_s \cos (\omega t + \epsilon_s)]^2\, dt = \frac{B_s^2}{2}$$

By equation (3) the relation between the mean squares of the output and input becomes

$$\overline{s^2} = \frac{\overline{F_s^2}}{|z_s(\omega)|^2} \quad (4)$$

The foregoing straightforward and elementary discussion assumed a pure cosine input. We set now the following problem: To find a relation between $\overline{s^2}$ and $\overline{F_s^2}$ if $F_s(t)$ is a random function. In other words, to find the mean-square amplitude of the ship motion, when the random input (the sea) is characterized by its statistical properties. It should be emphasized that $F_s(t)$ being a random function, one should not expect to be able to find $s(t)$. This seems to be important since some of the statements in the paper might be misinterpreted in this respect.

If $F_s(t)$ is a random function, we find its energy spectrum: $p_s(\omega)$ corresponding to the rather unfortunate $[r(\omega)]^2$ symbol in the paper. The method of finding $p_s(\omega)$ from the autocorrelation function is clearly described in the paper, therefore the only fact recalled here is that the mean square of a random function is the total area under the energy spectrum; i.e.,

$$\overline{F_s^2} = \int_0^\infty p_s(\omega)\, d\omega$$

and so

$$\overline{s^2} = \int_0^\infty q_s(\omega)\, d\omega \quad (5)$$

where $q_s(\omega)$ is the unknown energy spectrum of the output.

The theorem we now quote, without proof, says that for linear systems "the input and output energy spectra are connected the same way as the mean squares of the input and output" (equation 4)

$$q_s(\omega) = \frac{p_s(\omega)}{|z_s(\omega)|^2} \quad (6)$$

Substituting equation (6) in (5)

$$\overline{s^2} = \int_0^\infty \frac{p_s(\omega)}{|z_s(\omega)|^2}\, d\omega \quad (7)$$

It might be mentioned that the final result (equation 7), connecting the mean square of the ship motion with the impedance of the system and the energy spectrum of the waves, was obtained without assuming that either the input or the output ordinates are Gaussian.

2 The authors emphasize the importance of the energy spectrum of the sea, which in view of the preceding discussion is really the crucial point of the method. It should be mentioned, however, that while the energy spectrum is important for ship-motion studies, it does not give a plausible

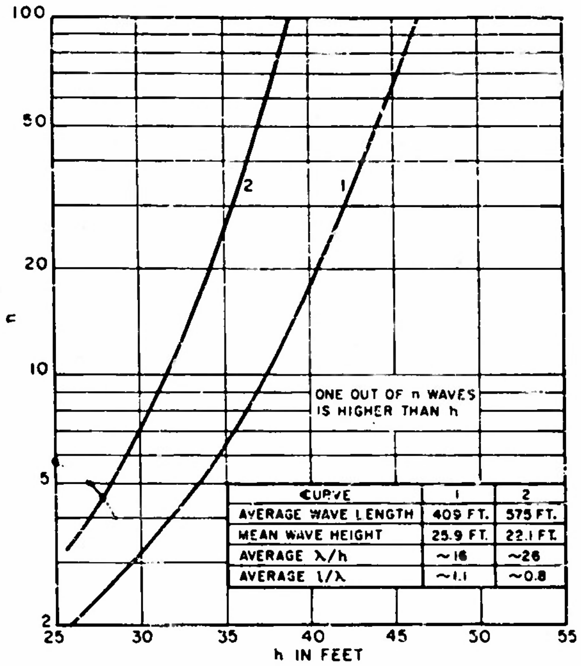

FIG. 1.—PROBABILITY CURVES ASSUMING NORMAL DISTRIBUTION OF WAVE AMPLITUDES

physical picture of the sea. The following discussion is intended to give such a physical picture.

Recently we performed experiments at the David Taylor Model Basin in the 140-ft basin to study the problem of slamming and for this purpose we produced a confused sea by generating regular waves with a pneumatic wavemaker and disturbing these with paddles and reflectors. The wave-height record was analyzed and showed that the amplitude distribution was not Gaussian but it definitely had a skewed tendency to the smaller heights (to the left). It might be mentioned that the paper is not entirely clear when speaking about the Gaussian property of the sea. Is the wave amplitude distribution or the distribution of the water surface elevation Gaussian? The writer must assume the latter possibility to be correct not knowing of any theoretical reasons or observations supporting the former one. The recent paper by A. J. Williams[22] suggests also skewed distribution for the wave heights, supporting this view by observation and theory.

A few numerical values might show the physical significance of the statistical approach. The model of a Liberty ship has an over-all length of 5.83 ft, that of the actual ship being $L = 441.5$ ft. The generated regular waves corresponded to a wave length of 545 ft, which after disturbance resulted in an average wave length of $\lambda_1 = 409$ ft and mean height of $h_1 = 25.9$ ft. Starting with a 606-ft regular wave the resulting average wave length and wave height were $\lambda_2 = 575$ ft and $h_2 = 22.1$ ft. The ship length/wave length ratios were $L/\lambda_1 = 1.08$ and $L/\lambda_2 = 0.77$. The standard deviations for the amplitudes were $\sigma_1 = 8.9$ ft and $\sigma_2 = 7.3$ ft and the dimensionless skewness factors $p_1 = -0.405$ and $p_2 = -0.58$.

If the skewness is disregarded and a Gaussian distribution is assumed, the probability that in the first case a wave will be higher than $h_1 + \sigma_1 = 34.8$ ft is 0.16; i.e., roughly speaking, one out of seven waves will be higher than 35 ft. In the second example approximately one out of seven waves will be higher than 30 ft.

The two examples were selected so that the ship length/wave length ratio was slightly larger than unity for the first case and somewhat smaller than unity for the second case. Undoubtedly these values and Fig. 1 of this discussion give a fair idea of the sea condition.

We might remark that both conditions correspond to the Scripps Institution's observation regarding the most probable relation for any given year for the North Pacific Ocean. In fact, both sea states could be generated by a 34-knot wind, the first requiring a duration of 12, the second of 24 hrs. The two examples of many experiments are presented to give a physical picture of an actually produced confused sea.

3 It is unfortunate that this excellent paper mentions hardly any applications and therefore it seems to be desirable to suggest at least one possibility of direct use of the paper. We attempt to indicate how one of the challenging problems of naval architecture, slamming of ships, might be handled by the methods proposed in the paper.

Either from wave observation or from theoretical consideration the wave-energy spectrum can be obtained and, making use of the response-amplitude operator and of the frequency mapping, the mean square, say of the pitching amplitude, can be obtained. Since large pitch angle alone does not result necessarily in slamming, other characteristics of the ship motion such as the mean square of the heave must be determined.

Assuming again a Gaussian distribution for the variables we arrive at a set of definite values for the probabilities that these variables will reach certain values. The multiple probability distribution of the foregoing variables will tell us the probability that a certain critical combination of the values of the variables will occur. The probability of slamming is not defined, of course, by the probability of extreme values of the parameters of the ship motion only, but also by the probability

[22] Quarterly Transactions, Institution of Naval Architects, vol. 95, 1953, pp. 70-90.

of having a surface elevation distribution favorable to slamming. The combination of these probability distributions, i.e., that of ship position and of surface elevation, will serve as a guide in slamming predictions. The writer would like to hear the authors' opinion regarding the proposition outlined.

DR. J. H. CHADWICK, *Associate Member:* While the writer is sure this paper is not intended to be the final word on the motions of ships in confused seas, undoubtedly it will stand as a land mark of the first magnitude in the history of the study of ship motions.

The authors have had to use some rather formidable mathematics in treating this problem, and some may feel at first glance that the mathematics is no more easy to understand than the seaway itself. However, this is not really the case, and it is predicted that in a very few years naval architects will be using these formulas with the same nonchalance that they now use the standard wave, and with infinitely more meaningful and useful results.

In this maiden voyage into confused seas, as it were, the authors undoubtedly have desired to keep the over-all presentation as simple as possible under the circumstances, and evidently it is this desire, coupled with the inherent complexity of the mathematics necessary to describe the wave system that has led them to use rather drastic approximations in deriving the equations of motion of the ship. They have assumed that yaw, surge, and sway are negligible, and they have neglected all couplings.

It is believable that, in a ship with a high-performance automatic steering system, yawing might be negligible; and, for small motions, surge also might be negligible compared to the forward speed of the ship. But in what sense can sway be negligible in a confused seaway; and since sway has a strong linear coupling into roll, in what sense can its effect be negligible? Furthermore, at speeds other than zero, pitch has a linear coupling into the heave equation, either through pitch rate (centrifugal force) or pitch angle (lift force) depending on the co-ordinate system used, and in what sense can this linear coupling be negligible? If yaw is not held down by an automatic steering system, it will couple into sway, and hence into roll, and roll will couple back into yaw giving rise to the so-called yaw-heel effects, and so on.

These points are mentioned in order to emphasize that, while the authors have shown us how to derive the statistics of ship motions from the statistics of wave motions, given the response operators of the ship, we still have a lot to learn about these response operators. Thus far it has not proved to be an easy matter to determine these response operators theoretically; i.e., for an actual ship with six degrees of freedom. Nor has it been easy to determine them by model tests; there is a host of difficulties that plague this kind of testing. While we should continue along both of these lines, is it possible that progress might not also be made through trying to solve the inverse problem; i.e., given the wave motions and the ship motions, what are the response operators, or, given a variety of ship motions, what are the couplings, and so on? Can the authors give us any hints about the possibilities in this direction, and about the mathematical methods that might be needed for the realization of these possibilities, if any?

MR. PAUL KAPLAN,[23] *Visitor:* This interesting paper is an application of the theory of random processes to the linear dynamics of a body in a fluid. The techniques used in the analysis also have been applied recently to dynamic problems of aircraft such as gust loads and buffeting, while the hydrodynamicist may be more familiar with their application to the theory of turbulence. With such a problem as the motion of ships in confused seas, the physical phenomena are so complex that detailed calculations of dynamic responses are difficult and the numerical results are uncertain. The physical data can be regarded as a set of statistical data of random processes; hence, the present statistical approach to this problem.

The main assumption of this paper is that the response of the ship in confused seas is a steady-state process. The response of a linear system to an arbitrary forcing function represented by a Fourier integral will be represented by another Fourier integral involving the transfer function of the system (response to sinusoidal forcing function of unit amplitude). When the forcing function is a stationary[24] random function, it does not tend to zero as $t \to \infty$ and its Fourier transform does not exist. Thus, another method must be used.

A way out of this dilemma has been found by the authors through an application of the results of Wiener's theory of "generalized harmonic analysis." The spectrum of the response is the product of the spectrum of the forcing function with the square of the absolute value of the system admittance. (Admittance is a term bor-

[23] Experimental Towing Tank, Stevens Institute of Technology, Hoboken, N.J.

[24] Mean values do not depend on time, but are functions of the time difference only. Also, time averages are equivalent to ensemble averages over all states of the ensemble.

rowed from electrical terminology. For a second-order system it is a complex quantity whose amplitude is proportional to the magnification factor μ_s.) In the present case, the result is the product of the wave spectrum by the square of the transfer function. Integration of the response spectrum over all frequencies is then the mean-square value of the response. Complications arise only in the modifications of this result by introducing the frequency of encounter and its associated mapping requirements.

The expression for the mean-square response of the ship described in the foregoing is also applicable to linear dynamical systems of higher order than the second-order system treated in the paper. Thus, if a linear system of equations could represent the coupled heave and pitch motions, the resultant fourth-order equation for either the pitch or heave response could be treated in the same manner as was done in the paper. The experimental response as measured in a towing tank, which includes the effect of the coupled motion, would yield the proper transfer functions and the mean-square responses could be determined in this manner also.

Since one of the aims of the study of ship dynamics is the determination of the accelerations that the ship will experience in a seaway, a statistical representation of the accelerations would be useful. The writer believes that this can be found easily by applying Wiener's theory. Since the system admittance or the transfer function for a displacement is determined for a sinusoidal forcing function of the form $e^{i\omega t}$ the corresponding quantities for an acceleration are just ω^2 times that for the displacement. Thus, the spectrum for the accelerations is just ω^4 times that for the displacements and the mean-square values of accelerations are found by integration. The various statistical properties of the accelerations then can be determined in the same manner as for the displacements and thus can be related to the loads that the ship structure will experience.

It is mentioned in the paper that the extreme values of the amplitude of motions can be predicted by the techniques developed by Longuet-Higgins. The probability distribution function and the most probable value of the largest value of a displacement in N-samples when N is very large are given by Longuet-Higgins in his paper on the height of sea waves. The distribution function is the same as that found in standard works on the statistical theory of extreme values and the expression for the extreme value is approximately the same. Since these results have been applied to problems such as gust loads on aircraft encountering atmospheric turbulence, it seems profitable to search the aircraft literature for further information to be applied to ship motions. After all, the ship and the airplane are just different types of bodies that are in motion in a semi-infinite fluid medium and atmospheric turbulence has properties of randomness that are similar to those of ocean waves.

It is an inherent property of the spectral representation of the response that phase relationships between the excitation and the response are obliterated in the analysis. This is unfortunate since phase relations appear to be the key to determining the wetness of the decks, the stresses and the added resistance in a seaway. For the irregular random functions that are dealt with in the paper, the concept of phase is rather difficult to formulate. Phase has meaning only in regard to periodic functions. What is of importance in the problem of ship motions in confused seas is to know the position and orientation of the ship relative to the irregular wave pattern. The relative position with respect to the waves can be determined by comparing the time histories of the motion and the wave simultaneously.

A method of determining the motion of a ship in irregular long-crested waves as a direct function of time (for the case of zero speed of advance and only head or following seas) has been developed in a recent report from the University of California. The motion is expressed as a convolution-type integral of the recorded surface-wave history and a derived kernel function. The kernel function is the Fourier transform of the ship response to a sinusoidal forcing function. Comparison of the theoretical results with the experimental motions showed very good agreement.

If this method could be extended to the case of a ship moving in waves at a particular speed and heading, then the motion in long-crested waves composed of two components at different headings could be determined. This would give the motions in a cross sea, for example. Of course, a record of the surface wave as a function of time when moving into the separate wave systems would be necessary. Whether it is possible to do this the writer cannot say, but the idea may result in something useful.

In conclusion, the writer wishes to congratulate the authors on their pioneer paper which introduces reality into the previously idealized treatment of ship dynamics. The results obtained are valuable in themselves as a first step and they indicate also the further work necessary for a complete knowledge of ship motions in a confused sea.

Dr. G. Weinblum, *Member:* Elaborate computations have shown that a Gaussian distribution describes reasonably well the actual state of a seaway. The Fourier integral method practically does not lend itself to the investigation of the seaway. This is an important result which will deeply influence research in the field of oceanography as well as naval architecture.

Fig. 5.1 of the paper shows that in a confused sea the pitching periods are shorter than the rolling periods. This known fact can be explained by the authors' theory in a more consistent way as by earlier orthodox ideas. This success alone justifies the great labor involved. The authors emphasize that phase relations are beyond the scope of the present investigation pointing out that phase conditions are of lesser importance in most practical applications. This statement may be true or erroneous; in any case the solution of the amplitude problem will be interesting enough if the authors succeed with their difficult task.

The bold extension of our knowledge concerning the seaway will stimulate further research in the field of motions and, more generally, seaworthiness of ships. Clearly, it would be a mistake to neglect the concept of regular seaway; on the contrary, stronger attempts must be made to elucidate further this basic case. Interesting and promising work in this field is now being done by Dr. Grim of Hamburg, which deserves careful consideration.

Mr. David Williams,[26] *Visitor:* The paper is of great interest to us because we are working on a problem which requires an analytical specification of ship motion. Part of this program has entailed the collection and analysis of records of motion of ships in their actual environment. Thus, we are in a position to offer partial confirmation of the conclusions reached in the paper.

In particular, we have tested the distribution of ship displacements, and have found in every case examined that the distribution was Gaussian. Furthermore, we have constructed a power spectrum analyzer for these records, with, we believe, very satisfactory characteristics at the low frequencies involved. The output of the analyzer, when plotted, failed to disclose any graininess, down to the limits of resolution of the filter, except for some peaks attributable to the elastic characteristics of the hull.

The motion of the ship as a rigid body in pitch had a relatively broad peak with a sharp cutoff characteristic at frequencies above the peak power portion. The roll characteristic was much more sharply peaked. No evidence was found of coupling between pitch and roll. As stated, the roll spectrum had a single very sharp peak; no peak appeared in the pitch spectrum at or near the same frequency.

These findings tend to support the hypothesis of a continuous spectrum of sea energy; at least the grid of a line spectrum would be so closely spaced as to be practically indistinguishable from a continuous spectrum.

Early in our investigation we formulated the concept of ship motion as white noise, originating in random wind disturbances and shaped by the successive filter characteristics of the sea and ship. Following this line of thought, we attempted to separate ship effects from sea effects.

David Taylor Model Basin, at our request, tested a model of a ship from which some motion records were taken, to determine its transfer characteristics. When these were divided out of the recorded power spectra, we found that the sea motion which resulted, was, within the broad limits of accuracy of our measurements, approximately white over a broad band containing all of the significant power in the ship motion.

We hope that this line will be followed vigorously both by mathematicians and experimenters.

Mr. Lawrence W. Ward, *Associate Member:* This paper is a striking example of the benefit of co-operation between different fields of science. In a few short years the sciences of meteorology and communications have gotten together to produce a simple and practical representation of the seaway, and the recent collaboration between the oceanographer and naval architect, only a year in being, shows the same promise in the field of ship motions.

On the experimental approach, the use of electronic computers may prove to be the best technique. Changes in the ship characteristics may be made easily by "adjusting a knob," and use of a random input is understood to be no more difficult than any other. It would be hard to simulate the latter conditions in a tank.

A change in terminology is noted in the paper which is not familiar to the writer; namely, the use of "response-amplitude operator," as opposed to "transfer function." In so far as the writer can determine the two are identical, and the latter is easier to say. Have the authors a definite reason for this change? Also, have the small boat tests at Jamestown indicated anything as yet as to the verification of the existence of this function?

The writer expresses his appreciation to the authors for their excellent paper, and may they continue to collaborate in the future.

[26] Bell Aircraft Corporation, Buffalo, N. Y.

MR. B. K. COUPER,[26] *Visitor:* The application of statistical methods to wave and motion studies seems to be full of promise. In addition, the writer would like to recall two fortunate occurrences and to express one sincere hope.

1 The recognition of the identity of wave motion and noise spectra has permitted the use of powerful tools developed in the electronics field to be applied to description of the ever-changing ocean surface.

2 It should be emphasized that this paper is the joint product of a naval architect and an oceanographer, of a designer and an environmentalist.

From such fortuitous combinations of fields of interest (and complementary talents) is woven the fabric of science.

The writer's hope is also twofold:

1 That continually improving instrumentation will permit additional verification of theory and more quantitative results.

2 That these quantitative results will include improved prediction of short-time, actual motions of a ship, as well as the prediction of "statistical parameters."

PROF. FRANK M. LEWIS, *Honorary Vice-President:* The application of statistical methods to the theory of ocean waves and the motion of ships is a new and interesting approach to the problem and may, in time, lead to fruitful results. But, in the paper, the authors hardly more than hint at possible applications and a more explicit statement of the ultimate objectives would add greatly to the value of the paper.

The limitation to linearity appears to offer a serious handicap in applications of the theory. For example, one might ask: What is the maximum stress that can occur from a given irregular sea? This would appear to be a simple matter. The sea record over a sufficiently long interval is analyzed into its harmonic components. Then the maximum wave could be merely the sum of all these components added crest on crest in zero phase. But this would hardly lead to a very satisfactory answer; for the wave height is limited by nonlinearity and the dissipation of energy of a breaking crest. It appears therefore that the maximum strain that a vessel can be subjected to under a given sea condition must remain a matter of long-time observation rather than of theory.

The meaning of the authors' statement in the conclusion that only amplitude and not the phase of the motion resulting from a confused sea can be obtained is not at all clear. On the basis of the authors' assumptions of linearity, the motion resulting from any arbitrary form of sea can be derived, although, to be sure, with considerable labor. One first has to derive the excitation corresponding to the irregular sea by placing the vessel in successive positions and integrating forces and moments. Then, in the authors' notation the equation of motion can be written

$$\frac{d^2s}{dt^2} + 2h_s\frac{ds}{dt} + \nu_s^2 s = f(t)$$

and this has the solution

$$x = \frac{1}{\alpha - \beta}\left[e^{\alpha t}\int_0^t f(t)e^{-\alpha t}dt - e^{\beta t}\int_0^t f(t)e^{-\beta t}\,dt\right] + F(t)$$

$$\alpha, \beta = -h_s \pm i(\nu_s^2 - h_s^2)$$

$F(t)$ is the damped exponential corresponding to the initial motion at $t = 0$.

Putting this in real form and dividing the range of integration into sufficient overlapping intervals, the foregoing expression can be integrated by mechanical methods. However, it would be easier and more accurate to utilize a model placed in an arbitrarily irregular sea.

It is not clear why an irregular sea will give better criteria of ship performance than a regular one. At intervals any irregular sea will become sufficiently regular so that the motion of the vessel, for a short time, will approach that attained in resonance. This is evident from an inspection of the authors' motion curves. Why is not this motion as "realistic" a criterion of ship performance as any other? A "realistic" irregular sea is at intervals, regular.

The authors touch briefly on the problem of short-time prediction, which, if we understand their statements, is as follows: Given the motion of the ship, and possibly also of the sea surface, from $t = -\infty$ to $t = 0$, or if not to $t = -\infty$ for a sufficient period in the past, can the motion of the ship be predicted for a short interval of positive t?

In attempting to answer this question, it is to be noted first that the surface of the sea must be considered as a purely arbitrary function, in which the waves approaching the ship (positive t) are related to those which have passed (negative t) only by the very loose condition of continuity of wave height and slope at $t = 0$. A harmonic analysis of past waves cannot be made and extended into the future, for there exists, an infinitude of such analyses, any of which would represent the past equally well, but would give entirely different results as regards the future.

[26] Bureau of Ships, Navy Department, Washington, D. C.

At $t = 0$ the existing motion of the ship is known and can be considered as a phase of a harmonic motion. One can predict that this existing motion will continue and that to it will be added a deviation produced by the approaching and unknown wave pattern. Unless this wave pattern can be foretold, the deviation must remain unknown. On the basis of statistical theory one also could predict that if the motion is large or small at $t = 0$ it will return to the mean, but, if already at the mean, one could not predict whether it would get larger or smaller. Since presumably the maximum, minimum, and mean motions are already known as a matter of observation, it is not clear how the statistical theory can assist.

One can ask, of course, whether a pattern does exist in the irregular sea, or whether it is as unpredictable as the spin of a perfect roulette wheel. The authors appear to have excluded the possibility of a pattern, and we concur in this view.

Unless then a device can be arranged which will reach ahead of the ship and relay back information as to the character of the approaching sea, only a low order of predictability of future motion is possible. A trained observer at the masthead might go far in this direction, although more elaborate and accurate devices are conceivable.

Prof. F. Ursell,[27] *Visitor:* The correlation of wave motion and ship motion is likely to be of value in oceanography, since it provides a simple method of measuring the energy spectrum of waves from the spectrum of the ship motion. But until more information is available about the response of a ship to regular waves, and in particular about the Froude-Krylov hypothesis, an assessment of the limitations inherent in the present paper will not be possible.

Unpublished experimental data suggest that the dependence of virtual mass and moment of inertia on frequency and speed of advance (neglected in the present paper) also may be important in this connection. Even so, a calculation within the framework of a linear theory may still be practicable, though it may prove to be inconveniently complicated.

Equations (1.2), (1.3), and (1.4) apply only to windless areas. Do the authors consider that this implies a serious restriction on the applicability of their work?

Mr. Norman H. Jasper, *Member:* This paper brings the methods of statistical analysis to bear on the problem of describing the sea state and the ship's motion in confused seas. It does this, however, in a manner so complex that naval architects perhaps will be more confused than enlightened by it. The writer feels that the presentation given by the authors is unnecessarily cumbersome and that the claims made for this method are too broad.

It should be emphasized that the statistical treatment preserves only certain qualities of the time functions, namely, the frequency and amplitude content (power spectrum), and such methods therefore cannot predict the actual motions but can only predict statistical qualities of such motions.

The problem treated here is essentially that of determining one (not all) of the properties of the sea: namely, its power spectrum; and next to determine the power spectrum of the ship's motion in this seaway. Among the several assumptions made the following may be the most critical: (*a*) The ship acts like a simple oscillator, which assumption is probably acceptable, and (*b*) the wave propagation is described by equation (1.1), an assumption that is valid only in special cases and which restricts the validity of equations (1.17) and (1.21.) Once the so-called response-amplitude operator, A_ω, for the ship is known the power spectrum of the ship motion is specified immediately and may be written down as in equations (4.3) and (4.13).

The power spectrum of the ship motion may be utilized to estimate certain parameters of ship motions such as average and "significant" amplitudes. One now may ask the question: Why use the cumbersome energy integral which the authors seem to feel is the one and only approach? There would appear to be other possibilities utilizing generally understood concepts which may be as valid (or invalid) as the proposed method. For example, consider the Fourier series expansion; it may be represented by a discrete spectrum such as indicated by the authors' Fig. 1.4. (It also may be converted to an "equivalent" continuous spectrum.[28])

Let

$$r(t) = \sum_{n=1}^{N} r_n \cos(\omega_n t + \epsilon_n)$$

where $r(t)$ represents a suitable section of an actual record. Next let ϵ_n assume random phases in the same manner that the authors assigned random phases to $[r^2(\omega)d\omega]$. This will result in an artificial ensemble of $r(t)$ each one of which is characterized by the same frequency or power

[27] Faculty of Mathematics, University of Cambridge, England.

[28] Naval Research Laboratory Report F3400.

spectrum. Therefore, by this approach we have

$$r(t) \cong \sum_{n=1}^{N} r_n \cos(\omega_n t + \epsilon_n)$$

this expression has the same form as equation (1.15).

A more generalized form of this approach can be obtained by using the Fourier integral equations (1.8) and (1.9). Let

$$r(t) = \int_0^\infty g(\omega)\cos(\omega t + \epsilon_\omega)\,d\omega$$

where

$$g(\omega) = \{a_p^2(\omega) + b_p^2(\omega)\}^{1/2}$$

$$a_p(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} r(t)\cos\omega t\,dt$$

$$b_p(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} r(t)\sin\omega t\,dt$$

In the expression for $r(t)$ let ϵ_ω take on random values. Thus, we can create an ensemble of $r(t)$ each one of which has the same power spectrum $[g(\omega)]^2$. If the power spectrum of the sea approaches the Gaussian form, $r(t)$ may be expected to have the statistical properties of a Gaussian distribution. The principal advantage of the approach outlined here is that the concept of Fourier series and of a Fourier integral are widely known.

Another relatively simple way in which the power spectra and other statistical parameters can be obtained is outlined by H. W. Liepmann.[29] Liepmann obtains the desired power spectrum from the Fourier transform of the autocorrelation function of the time series under study. Another statistical approach to the study of irregular seas has been given by R. Miche[30] who characterizes the waves by three parameters.

One might well ask in what specific applications these "predictions" might be used. Although the authors claim that it is their interest to present findings useful to the designer and researcher little is said in the paper about specific applications. Would a skipper be happier with a sea state specified by a power spectrum, even though this would have quantitative meaning, rather than with the present scale of sea states?

It is this writer's impression that equations such as (4.5) which describe a typical motion as a function of time are of little practical value and of doubtful validity. A rather useful and significant application of these methods could be made to determine the frequency distributions (histograms) of ships' motions and wave heights for a given combination of sea condition, ship, course, and speed.

MR. ROBERT A. FUCHS,[31] *Visitor:* The authors have rendered valuable service in carrying through the calculations of the frequency mapping and ship responses in such detail.

The difficulties discussed by the authors in connection with a Fourier integral representation can be largely avoided by adopting a somewhat different viewpoint. In the course of recent research sponsored by the U. S. Navy Bureau of Ships, David Taylor Model Basin, Contract Non-r-222(18) we have had occasion to analyze the motion of unpropelled ships in irregular long-crested waves by Fourier integral methods.[32]

We observe that the amplitude spectrum of the waves is not of direct interest in itself and since it is involved in both the oscillations of sea surface and ship it can be eliminated. We obtain, thereby, the motion of the ship expressed as a convolution-type integral of the product of the recorded wave motion and a "kernel" function. This "kernel" function is the Fourier integral, with respect to frequency, of the response of the ship to sinusoidal waves of unit amplitude.

The principal advantage of this formulation over a direct Fourier analysis is that the "kernel" is independent of the particular wave system encountered. Predictions of heaving and pitching motions of a rectangular block and a ship model in irregular waves agree fairly well with motions recorded in the laboratory wave channel.

The Froude-Krylov hypothesis was used to compute the response of the rectangular block. Two methods were employed to determine the response for the ship model. First, a series of tests was run in sinusoidal waves and the response measured directly; and then a Fourier analysis was made of the motion of a ship in a very irregular wave motion. Similar results are possible for propelled ships in short-crested seas, but the calculations involved are quite lengthy. Moreover, one requires for the predictions a surface-time history, not just at a single point, but over an extended area.

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[29] "On the Application of Statistical Concepts to the Buffeting Problem," by H. W. Liepmann, *Journal of the Aeronautical Sciences*, Dec. 1952.

[30] "Principal Statistical Criteria Characterizing Real Irregular Sea Waves and the Methods of Producing These Waves in the Laboratory" (in French), by R. Miche, Rev. Gen. Hyd. No. 69, 115-124, May-June, 1952.

[31] Institute of Engineering Research, University of California, Berkeley, Cal.

[32] "A Linear Theory of Ship Motion in Irregular Waves," by R. A. Fuchs and R. C. MacCamy, University of California IER Series 61, Issue 2, July, 1953.

It is hoped that the authors will in the near future present numerical applications of their methods to ships in confused seas. There appears to be a real need for detailed descriptions of the sea surface over an area as a function of time. One should not abandon, however, the possibility of establishing certain simple rules by studying incidents of a sufficiently simple nature. An important restriction, in so far as computation of the responses of the ship is concerned, is the acceptance of the Froude-Krylov hypothesis. This hypothesis is stated by the authors in an unnecessarily restricted form. The ship acts on the waves by virtue of the virtual masses, virtual moments of inertia, and damping terms, but wave diffraction usually is neglected. Since this diffraction is of considerable importance, not only for the ship oscillations, but also for resistance in a seaway, it appears to warrant more study than it has received heretofore.

MR. GERHARD NEUMANN,[23] *Visitor:* Wave research, especially with reference to the behavior of the seaway under various conditions, has been sponsored and greatly stimulated during the past ten years by those who apply the theory in practical applications. Navigation, naval architecture, marine and coastal engineering, and kindred practical professions are interested increasingly in the behavior of ocean waves and their effects.

The concept of the "confused" sea may really confuse those who are accustomed to applying this word for very special conditions of the seaway. It is true that even the pure wind-generated and wind-driven ocean waves appear very irregular with the everchanging pattern of their apparent waves. The heights and time intervals between successive crests ("periods") are the observable properties of the composite sea. They change from "wave" to "wave" and form the characteristic irregular pattern of the sea. This characteristic pattern is exactly the one in which the navigator, naval architect, or marine engineer is interested; and this pattern is closely related to the energy spectrum of the complex ocean waves.

Experienced observers know how to estimate the wind velocity from the appearance of the wind-driven sea with an accuracy of ½ deg of the Beaufort scale, and this shows that there must be a clearly defined dominating pattern of the seaway at each wind velocity. The dominating pattern, or the characteristics of the seaway, at each wind speed and given fetch and duration conditions can be deduced from the energy spectrum of the wave motions. Thus these conditions are not really confused.

A confused sea, however, may be, for example, the cross sea which forms with wind jumps after the passage of meteorological fronts or in the eye of a hurricane. With these conditions, seas from different directions and of different size cross each other and form, in storm regions, very dangerous pyramidal seas. The crests of these "waves" may break in either direction. This kind of confused sea is not meant in general in the present paper, and it is not described by the energy spectrum as derived by the writer of this comment. There is some hope that in the near future oceanographers will be able to give more information to naval architects on how to handle even such conditions of cross seas.

The experimenter in naval architecture needs some criterion as to how complex the tank waves must be in order to create an artificial wave motion comparable with the actual ocean waves at given wind conditions. This kind of information can be given by the use of the mathematical form of the energy spectrum, and approximation by a finite number of equivalent wave components. By the laws of similarity the absolute dimensions of the waves can be reduced to tank waves, according to the dimensions of the ship model, whereas the relative dimensions of the wave components are fixed by the wave spectrum at the given wind velocity.

The objection, that the concept of the wave spectrum does not give a plausible physical picture of the sea, is not at all justified. This objection, certainly, is a misunderstanding. Any kind of practical information on the dimensions and the statistical distribution of the heights, "periods," and "wave lengths" of successive apparent wave crests at a fixed location at the sea surface can be derived on the basis of the energy spectrum. A paper on the characteristics of the seaway in terms of simple wave properties is in preparation and will help the practical application.

The writer would like to add a few comments about the understandable demand by practitioners for more information regarding the actual sea spectra likely to be met under various conditions, and on different routes. This is the problem of wave forecasting. In some comments, and by personal communication to the writer, the hope was expressed that oceanographers can fill this gap. It must be confessed by oceanographers that there are many open questions in ocean-wave research and that much more work still has to be done in the field of theory and observation before

[23] College of Engineering, Department of Meteorology and Oceanography, New York University, New York, N. Y.

all the problems about ocean waves can be answered satisfactorily.

The new methods for forecasting the actual state of the sea at given wind speeds, fetches and durations of wind action, as given and explained in several recent publications, are a step forward in this direction. References [12] and [15] in the paper give the necessary information. They permit the forecasting of the wave spectra for different weather conditions and on different routes.

MR. JOHN W. TUKEY,[34] *Visitor:* It is a great pleasure to see the techniques developed for dealing with "noise" (acoustical or electrical functions of the time which are "confused," showing both randomness and distinctive characteristics) spreading through the study of ocean waves to ship's motion, and thus, no doubt, to the strains, stresses and accelerations of the various parts of the ship. By analogy with other fields, we can expect both new insight and new conclusions to follow. The writer predicts this will prove to be an important pioneering paper.

Formula (1.10) appears very strange at first sight. (This strangeness seems to have very considerable value in causing the reader to stop and think.) This strangeness should not, it is believed, be associated with the name of Lebesque. The approximating sums indicated in formula (1.14) are the same type of sums as occur in ordinary (Riemann) integration. The strangeness may, however, be associated properly with the nature of the approximating sums and the nature of their limit.

The authors do not seem to have made it clear that equation (1.10) represents not one sea surface history but a whole ensemble of sea-surface histories. Here ensemble is used in the sense of statistical mechanics and indicates an assignment of probability to the functions of t which describe possible sea-surface history. The situation is entirely analogous to the simpler one where a statistical distribution assigns probabilities to various possible values of a number or a vector. The authors really consider the (linearized) response of a ship to all possible sea-surface histories, giving appropriate probabilities to the resulting histories of ship's motion. The usefulness of this method of analysis is much enhanced by the fact that all histories of the ship's motion and of the sea surface are considered.

A simple example may clarify the situation somewhat. Let $\eta(u)$ take on the values of ± 1 with equal probability for every u, and let each $\eta(u)$ for each value of u be independent of the $\eta(u)$ for all other values of u. Consider

$$\int_0^1 \eta(u)\sqrt{du}$$

The same approximating sums are (for 1, 2, and 4 equal intervals)

$$\sqrt{1}\,\eta\left(\frac{1}{2}\right) = \begin{cases} -1, & \text{probability } 1/2 \\ +1, & \text{probability } 1/2 \end{cases}$$

$$\sqrt{1/2}\,\eta\left(\frac{1}{4}\right) + \sqrt{1/2}\,\eta\left(\frac{3}{4}\right) = \begin{cases} -\sqrt{2}, & \text{probability } 1/4 \\ 0, & \text{probability } 1/2 \\ \sqrt{2}, & \text{probability } 1/4 \end{cases}$$

$$\frac{1}{2}\eta\left(\frac{1}{8}\right) + \frac{1}{2}\eta\left(\frac{3}{8}\right) + \frac{1}{2}\eta\left(\frac{5}{8}\right) + \frac{1}{2}\eta\left(\frac{7}{8}\right) = \begin{cases} -2, & \text{probability } 1/16 \\ -1, & \text{probability } 4/16 \\ 0, & \text{probability } 6/16 \\ 1, & \text{probability } 4/16 \\ 2, & \text{probability } 1/16 \end{cases}$$

The limit of these approximating sums, each of which is a distribution, is a Gaussian distribution with mean zero and unit variance. It is easy to see that replacing $\sqrt{du}$ by du gives a limiting distribution involving only the value zero. The square root is essential.

As a second example, consider

$$\int_0^1 \cos\left(t + \epsilon(u)\right)\sqrt{du}$$

where, as in (1.14), each $\epsilon(u)$ is uniformly distributed on $[0, 2\pi]$ independently of all others. When the approximating sums are calculated, each will be an ensemble, and each will be of the form

$$[A \cos (t + \epsilon)]$$

where $A \geq 0$ and ϵ is uniformly distributed on $[0, 2\pi]$ independently of A. The distribution of A will depend on the number of intervals in the ap-

[34] Bell Telephone Laboratories, Inc., Murray Hill, N. J.

proximating sum, being concentrated at $+1$ for one interval and converging to the distribution of $B^2 + C^2$, where B and C are independently distributed Gaussian distributions with mean zero and variance $\frac{1}{2}$. (This follows easily by considering $A \cos (t + \epsilon) = B \cos t + C \sin t$). Again replacing $\sqrt{du}$ by du would make the integral identically zero.

In this example the ensemble consists only of sinusoidal functions of the time, of period 2π, and can be easily described verbally or by simple formulas. However, when we reach the level of complexity required for the representation of a confused sea, the description given by (1.10) is probably the simplest precise formulation as well as the one easiest to manipulate. The square root is still essential.

Each of the authors' approximating sums in (1.14), where we take the ω_i as fixed and the $\epsilon(\omega_i)$ as independently and uniformly distributed according to (1.13), is an ensemble of functions of t. Their limit is also an ensemble of functions of t. (The idea of working with ensembles is unfamiliar, but very useful and powerful.)

Finally, the writer should like to suggest that the authors' words of warning concerning the difficulties of treating phases need not be taken too seriously. The analysis will become more complex, but not by a factor of more than 2 or 3. When such analyses are needed, they will be performed, and without serious difficulty. Moreover, recent unpublished developments suggest that the questions of nonlinearity which the authors allude to in passing also can be handled when really necessary. The increase in complexity will be considerable but solutions can reasonably be expected. The writer agrees with the authors, however, that the linear (and hence Gaussian) analysis which they have given will suffice for most purposes so far in view.

MR. WILBUR MARKS,[35] *Visitor:* One of the branches of oceanography is concerned with the study of the configuration of the sea surface and the prediction processes associated with the propagation of wind-generated surface waves. The oceanographer has suffered from a simple case of not seeing the forest because of the trees. That is, the tendency has been to study in great detail the phenomenon known as the seaway without any regard to that system which is most affected by it, seagoing vessels. On the other hand, the naval architect plans and builds his ships, always keeping in mind that the seaway has a great effect on his vessel and always trying to compensate for this effect, yet never quite understanding the full implications of the properties of the seaway.

The first joint effort of the naval architect and oceanographer has presented a clear picture of the very complex sea surface in a way that suggests the feasibility of the use of the seaway as a tool for the naval architect rather than a mystical phenomenon heretofore considered as a formidable foe.

The paper demonstrates the way in which a knowledge of the configuration of the sea surface (based on energy considerations) combined with the response-amplitude operators (which depend on the properties of the vessel) and a detailed frequency mapping can be used to determine the amplitude responses of the vessel in three chosen degrees of freedom. The limitations on the theory are stated clearly so that the boundaries of its sweeping conclusions are clearly defined to even the casual reader.

No theoretician is ever quite satisfied with the results of his efforts unless the conclusions are verified by experiment. Here again, a joint effort has been undertaken by the naval architect and oceanographer. At the Woods Hole Oceanographic Institution, a 35-ft launch has been carefully outfitted with devices designed to record roll, pitch, and heave. These motions are being studied in a seaway composed of waves 2 to 3 ft high (which eliminates the nonlinear effects of high waves), the whole system being analogous to a large ship in a more energetic seaway. The configuration of the sea surface is measured from stereophotographs taken from a bridge situated 130 ft above the seaway. The response-amplitude operators are computed from the theoretical considerations discussed in the paper. The ship responses also are measured so that the answer to the problem is known beforehand and thus may serve as a check on the computed response operators.

The information so obtained makes it possible to evaluate the energy spectrum of the sea surface and of the ship motions by the methods outlined by Marks (reference [11] of the paper). If the procedure of this paper is followed, the product of the energy spectrum of the seaway with the response-amplitude operators and the proper frequency mapping will result in the energy spectrum of the vessel's motions. Since this is known, the solution is available as a check on the product above.

Preliminary investigation has yielded the ship responses in head and following seas for the vessel traveling at 6 knots. This is shown in Figs.

[35] Research Associate in Mathematics, Woods Hole Oceanographic Institution, Woods Hole, Mass.

2 and 3, herewith. It is interesting to note that the wind which generated the $2\frac{1}{2}$-ft waves, which produced the motions, was approximately 35 mph. The fetch is approximately 1 mile long, and it is necessary to have a fetch of at least 400 miles before a fully developed sea can be attained. The sea spectrum will have a maximum of $[r(\omega)]^2$ which is very much lower than that for the fully developed sea. For this fetch, even very high winds probably would have difficulty generating waves 4 ft high.

The motion records are similar to actual wave records and indeed have similar statistical characteristics (see section of the paper, "The Ship Motions"). Consequently, these records are treated in the same statistical manner as wave records (equations 5.2 to 5.10). The records of the ship motions for pitch and roll in Figs. 2 and 3,[36] herewith, are described by the second and third equations of (5.1). The motion of translation in the z-direction is measured by an accelerometer and the record may be represented as

$$(\ddot{z}t) = \int_0^\infty \cos\,[\omega_e t + \epsilon(\omega_e)]\sqrt{[\ddot{z}(\omega_e)]^2\,d\omega_e} \ldots (1)$$

In order to convert the acceleration record to one of heave, it must be integrated twice with respect to time. However, if the heave record equation (5.1) of the paper is differentiated twice with respect to time and set equal to equation (1) of this discussion, it is seen that a relationship exists between the heave spectrum $[z(\omega_e)]^2$ and the acceleration spectrum $[\ddot{z}(\omega_e)]^2$ that is

$$[\ddot{z}(\omega_e)]^2 = \omega_e^4[z(\omega_e)]^2 \ldots\ldots\ldots (2)$$

Consequently, if the spectrum of vertical acceleration is computed and the $[\ddot{z}(\omega_e)]^2$ at discrete values of ω_e are divided by the appropriate ω_e^4, the result is $[z(\omega_e)]^2$.

For head seas, the cumulative response amplitude densities of the motions R^*_ψ, R^*_φ, and R^*_z, are all greater than the corresponding values for following seas as comparison of Fig. 1 with Fig. 2 of this discussion shows. In addition, the frequencies for the motions in head seas, f_ψ and f_φ, are greater than for following seas. The result of this is that in head seas the product of the two-dimensional wave spectrum with the response-amplitude operators is such that significant amounts of energy in the waves are located at resonant frequencies of the response-amplitude operators and it is this which causes the large amplitude motions in head seas.

In following seas, the vessel encounters the significant amounts of energy in the seaway at much lower frequencies (higher periods of encounter), since the wave lengths are of the same order of magnitude as the length of the vessel. The pitch records show little relative difference in wave speed and ship speed and, hence, the vessel tips relatively slowly fore and aft as she adjusts to the profile under the ship.

In both head and following seas there are some spectral components which encountered the vessel from the quarter, the frequencies of encounter being different than in the case of encounter perpendicular to the crests of those waves. The result is an induced roll which produces higher roll amplitudes in head seas and higher roll periods in following seas.

[36] Figs. 2 and 3 of this discussion appear on page 357.

CAPT. HAROLD E. SAUNDERS, U.S.N., (Ret.) *Member*: This paper treats of a very important subject, and I should like very much to present a technical discussion of it, if only I could understand it.

For those who wish to learn about some of the fundamental concepts involved, I commend the report listed as reference [15] of the paper, by Dr. Pierson, Dr. Neumann, and Mr. James. This gives, in beautifully simple and straightforward language, much of the story necessary for an understanding of the present paper.

For all those members of the Society who have to write papers of this kind, I ask the reader's indulgence to quote a rather free translation of some remarks made by Admiral Barrillon on a paper read by an Italian engineer before the French Society of Naval Architects (Association Technique Maritime et Aéronautique, Proceedings, volume 33, 1929, page 72):

"... the paper may be read from one end to the other without the necessity of following the calculations, pen in hand, and without ever going back to look up the significance of a letter or a symbol. The reader comes to the end with the realization that nothing is forgotten and that an identical fundamental treatment, presented with integrals, would have added nothing, either in vigor or in generality. Last but not least, the reader is convinced that, when the day comes that he will want to try an application of this method, he will not have to relearn the calculations himself, line by line. On the contrary, he will be able to give the numerical work directly to a draftsman, to be done without special preparation."

MESSRS. ST. DENIS and PIERSON. The authors are grateful for the heartening response from so many distinguished researchers in this fascinating field of ship motions and would like to make the following rather incomplete closing remarks.

As to Professor Korvin-Kroukovsky's remark that we gave no estimate of the range of angles to be expected in practice, we should like to emphasize that our intent was to provide a method of analysis which would be valid regardless of the amount of short-crestedness. An idea as to the angular variation of the energy spectrum for a sea may be obtained from the theoretical spectrum of equation (1.28). This empirical equation is substantiated by a rather large mass of indirect evidence. No direct evidence is available because two-variables spectra of the sea surface have not as yet been obtained.

In regard to the comment on the necessity of considering coupled motions, reference is made to the third paragraph of the discussion by Mr. Chadwick wherein our reasons for restricting the paper to uncoupled motions are clearly brought out. For coupled motions the mathematics will be somewhat more complex, but the principles and the method will remain the same. The purpose in writing this paper was not to inquire how well the problem of ship motions in confused seas could be solved, but whether it could be solved at all. The logical extension to coupled motions will be carried out in the near future in connection with the experimental work under way at Woods Hole.

It is not possible to reply to the comment on the seriousness of the restriction to linearity on the basis of available data. Analyses of ship motions carried out so far indicate that the nonlinear effects may not be too serious. The vessels for which motions were analyzed comprised a troop transport, an aircraft carrier, and a motor launch. As concerns the analysis of the nonlinear effects the comment by Dr. Tukey is indeed promising.

We appreciate the comments of Mr. Cummins whose summary of possible applications answers questions raised by other discussers. The remark on the two types of nonlinearities is especially important.

We should like to reply to the criticism of Mr. Lewis (and incidentally also of Professor Korvin-Kroukovsky) that experimental methods were not given full justice, by noting that these are fairly obvious, rather simple, and quite well known. The intent was to present a logical alternate method.

We regret to disagree that head-on or following seas are the most important. This needs first to be established. For wetness it would appear that a sea having beam components inducing roll might well be of greater severity.

With reference to the availability of oceanographic data on wave characteristics, more up-to-date and complete results have been obtained since references by Kempf and Kent. These new results agree well with Neumann's theory presented in the paper. It is possible today to describe the energy spectrum of the seaway quite accurately from meteorological data.

The warning that the spectrum given in Fig. 1.6 should be used with caution is, we feel, not too well taken. In the trade-wind regions, for example, such a spectrum might be expected to exist for days on end, and it should represent the seaway over areas of thousands of square miles. Even for the case where the spectrum is building up, it can be forecasted by the techniques given in reference [15]. The constant wind blowing for a long time over an initially calm surface is sufficiently accurate an approximation to permit a description of the seaway for about half of the area of the North Atlantic Ocean most of the time. As soon as the wind ceases over a given area, the various waves in that area travel away so fast that the more complex spectra which are possible are rather rare phenomena which at the present time need not worry the investigator too much.

Of course when the wind shifts suddenly, cross seas develop which may be very severe. But shortly thereafter the seaway may well assume the form given by the theoretical spectrum.

The assumption that a complex cross sea can be considered as being made up of two long-crested seas coming from different directions is very attractive and would result in a considerable simplification of the problem. Unfortunately, there is insufficient knowledge in hand to determine under what conditions such a simplification would be reasonable.

The idea of using effective wave lengths is very good in so far as it aims to make the most use of presently available facilities and in so far as it is restricted to the motions of heave and pitch. The effect in roll cannot be considered in this manner.

We are looking forward with interest to the completion of the work at Woods Hole and will study it deeply when available. We hope that Nature and Dr. Iselin will conspire to bring about a verification of the theory presented in the paper.

The comments of Dr. Davidson are concise, precise, and to the point. He also is concerned with simplifying the problem of the confused sea by introducing a minimum number of components. Such a simplification with model studies should provide information on the validity of the assumption of linear superposition if the individual components are not too high. It should be emphasized, however, that going to an infinite number of components does not proportionally increase the complexity of the work. Indeed, the infinitely

complex (short-crested Gaussian seaway) can be treated with relatively little increase in effort over the simpler models.

The derivation given by Dr. Szebehely is valid for the heaving and pitching motions of a vessel in long-crested head and following seas and for the rolling motion of a vessel in beam seas. The extension of this method to short-crested seas will involve the same procedure as given in the paper.

With reference to the point that in the derivation of Dr. Szebehely it was not necessary to assume that the wave ordinates were Gaussian, it should be pointed out that a Gaussian function is a random function but that the inverse is not necessarily true. Waves are known to approximate closely Gaussian functions. This is fortunate since much more is known about Gaussian functions than about functions that are non-Gaussian. The assumption that the wave ordinates are indeed Gaussian functions permits a larger number of conclusions to be drawn than otherwise would be the case.

The criticism that $F_s(t)$ being a random function one should not expect to be able to find $s(t)$ is correct in the sense that the exact form of $s(t)$ cannot be derived. However the statistical properties of $F_s(t)$ and $s(t)$ are linked. Since the seaway is a stationary Gaussian process, we will never be able to do any better in a practical sense.

The generation of Gaussian seas in a laboratory is not a simple process and the paddle method described appears to be rather optimistic as an attempt to reproduce waves generated by nature.

In reply to the question as to whether wave amplitudes or wave ordinates follow a Gaussian distribution, it is, of course, the latter as brought out following Equation (1.16) in the test.

As to the second point, namely, that the representation of a sea by means of its energy spectrum is not a plausible one, we concur if this is interpreted to mean that the energy spectrum does not give a complete and accurate physical portrayal of the sea. Such a representation, however, does provide one with a knowledge of the essential statistical properties of the seaway. This is all that is required to obtain a statistical idea of the resulting ship motions. And since for a confused sea the motions can only be stated statistically, a knowledge of the energy spectrum is sufficient. Anything beyond that is useless.

The suggested outline for solving the problem of slamming appears to be reasonable. In addition phase relationships between excitation and response need to be investigated.

The kind remarks of Dr. Chadwick are encouraging. With reference to his comments on coupling it may be said that this problem can be tackled with hope of success as pointed out, for example, by Mr. Kaplan. The assumption that we endeavored to keep the presentation simple is correct. The equations of the coupled motions of a ship are formidable indeed. Fortunately, electronic computers are available along with powerful mathematical techniques; one should anticipate that present difficulties will be surmounted eventually.

The reply to the question on inverting the procedure is in the affirmative. If the ship motions and the spectrum of the seaway are known, the response-amplitude operators can be found. The technique involves running a model or a ship at a constant velocity at many different headings in the seaway so as to sort out the effects of varying frequencies of encounter.

The restatement by Mr. Kaplan of the authors' procedure is correct. As to the remark on phase relations attention is invited to two papers; the first by Press and Mazelsky,[37] the second by Press and Houbolt.[38]

Dr. Weinblum warns against neglecting the regular seaway. It is believed that by use of the approach presented in this paper the importance of studies of ship motions in regular seaways will be increased because the results will find an interpretation in the case when the seas are confused.

In reply to Mr. Williams, we note with satisfaction that independent observations have shown that various ship motions are Gaussian, that a partial confirmation of these theories is possible, and that an analysis of the spectra of these motions has not indicated any pronounced effect of coupling. These results suggest that a derivation which neglects coupling may prove to be a rather satisfactory approximation in the study of actual ship motions.

In reply to Mr. Ward, the terminology "transfer function" generally used in this connection was dropped in favor of the more cumbersome response-amplitude operator for reasons of rigor. The transfer function is used in operational calculus to determine the response of a system subject to a set of initial conditions. The response-amplitude operator is independent of the initial conditions and gives only the amplitude of the response, not the response itself.

The tests at Jamestown have just been concluded but the task of reducing and interpreting the data is still ahead. We look forward to the results with keen anticipation.

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[37] Press and Mazelsky [1953]: "A Study of the Application of Power-Spectral Methods of Generalized Harmonic Analysis to Gust Loads on Airplanes." NACA Tech. Note 2853, 1953.

[38] Press and Houbolt [1954]: "Some Applications of Generalized Harmonic Analysis to Gust Loads on Airplanes." Preprint #449, Institute of the Aeronautical Sciences.

Mr. Couper is well aware of the importance of oceanography in ship design. His comments serve to illustrate the thesis that mathematics is universal and that in mathematics what has been developed in one field can often be carried over to another field (except for minor changes in terminology) and applied with promising results. In the present paper this was done by taking results derived in electronic theory and extending them to ship motions.

In reply to Professor Lewis it should be emphasized that the scope of the paper was limited to an inquiry as to whether or not it were at all feasible to deal with the problem of ship motions in confused seas. Within the permissible length of presentation only the essential features of a theory could be given. Possible applications of the theory were discussed by Mr. Cummins.

Professor Lewis' suggested technique for obtaining the maximum possible stress does not work. When such an analysis is carried out, either on a wave or on a stress record, the longer the record the greater the number of components. If the amplitudes of all the individual components were to be combined in phase their sum would increase with length of record! The final result would be that any possible maximum stress could be obtained from the analysis by simply stretching or shrinking the stress record. The essence of the theory presented in the paper is that the components of wave or ship motion (or stress for that matter) cannot be combined in zero phase, but necessarily must be summed up in random phase. Certainly long before extremely high amplitudes could be obtained the problem becomes nonlinear and the results are inapplicable. Reference is made again to the paper by Press and Mazelsky[36] wherein the same techniques have been applied to stresses in aircraft structures with remarkable results. Once it is recognized that the problem is essentially statistical, it is possible to advance.

Phase relations were not considered in the paper. A complete theory, of course, should take them into account. As pointed out by Professor Tukey, it is possible to do so but at the expense of an increase in complexity.

We regret that we cannot understand the argument that irregular seas are regular at times. The fact that on certain days a sea can be dead calm does not affect the motions of a ship in a seaway caused by a gale. The argument is rendered more obscure by Professor Lewis' concurrence as to the absence of an (analytical) pattern in an irregular sea.

Although the problem of ship motions in a confused sea can be approached through a method based on transients, it is exceedingly doubtful that this can lead to a practical solution. One of the powerful advantages of the technique presented in the paper is that ship motions can be treated statistically as steady-state phenomena and not as a succession of transients. The analysis by numerical methods of the exact motion of a ship in a complex seaway for a reasonably long time (20 min) is an extremely cumbersome task even with a large staff and high-speed computers.

Perhaps the concept of prediction should be clarified. Short range predictions of ship motions (5 to 10 sec) can be made solely on the basis of the past history of the motions. This is accomplished by means of the prediction theory developed by Wiener [26]. The method depends only upon the autocorrelation function of the record with itself. Of course, the prediction becomes progressively worse as it is extended in the future. In this case the form of the seaway has nothing to do with the problem.

Does a confused seaway have a pattern? In an analytical sense the answer is probably no. A pattern is not excluded, but is improbable. In a statistical sense the answer is definitely yes. It is actually the existence of such a statistical pattern that leads to practical results.

The idea of a device for foretelling the character of the sea approaching the ship is being investigated at present with the intent of improving predictions. The improvement should result from correlating the seaway not only in time but also in space.

We regret to disagree with Professor Ursell that we have provided a simple method for obtaining the energy spectrum of the waves from that of the ship. The wave spectrum is highly distorted by the mapping and different wave-spectral frequencies can give rise to the same frequency of encounter. It is possible to determine uniquely a ship motion from a known seaway. The inverse, however, does not hold for, given a motion, the seaway is not uniquely derivable.

There is no doubt whatsoever that the validity of the Froude-Krylov hypothesis must be examined critically. The dependency of the inertia and damping coefficients upon speed of advance also must be determined. These are logical next steps.

The wave formulas, equations (1.2, 1.3, 1.4), given in the paper do not hold in Nature even for windless areas since even a swell is Gaussian. Observations indicate that seaways constructed from these formulas as in Equation [1.21] agree remarkably well with reality.

The effect of wind on ship motions has been neglected as constituting a nonessential complication. The wind effect on ship motions can be

dealt with in a similar manner as the wave effect, introducing, however, the above water body of the ship. The method, in this case, is highly empirical. There is some simplification arising from the single direction that the wind has at any given point and instant.

The first criticism of Mr. Jasper on the complexity of the method presented has been answered by Dr. Chadwick. When an idea is new to an audience, it has to be elaborated somewhat more fully and with a greater degree of rigor than would otherwise be the case.

It is true that a statistical method cannot predict actual motions, but only certain properties thereof. Except for short-range predictions in the sense of Wiener, this is all that will ever be possible with a seaway or a ship. That these properties are all derivable from the energy spectrum is no handicap; any significant property of the seaway can be determined from the energy spectrum and such properties as cannot be determined from the energy spectrum are not significant.

The question was raised of the applicability of equation (1.1) and, consequently, of (1.17) and (1.21). These all rest on the assumption of linearity. There is mounting evidence that the last equation is a highly valid representation of the seaway. In support of this may be cited a recent work by Cox and Munk[39] giving results substantiating Neumann's theoretical spectrum.

With respect to the objections to the cumbersome energy integrals, reference is made to the comments by Professor Tukey. The alternate models are valid up to a point, but their extension to short-crested seaways is difficult. The integral representation in terms of $g(\omega)$ suggested by Mr. Jasper is invalid. When $\epsilon(\omega)$ takes on random values, an integral of the suggested form has a value identically zero as pointed out in the discussion of Professor Tukey.

It is not the energy spectrum of the seaway which has a Gaussian form; it is the seaway record that is Gaussian. The energy spectrum may have any functional form. In so far as it can be determined at present the Neumann spectrum gives such a form for a wind-generated seaway. A filtered portion of this spectrum gives the functional form for a swell.

The technique, described by Mr. Jasper, for determining the energy spectrum from the Fourier transform of the autocorrelation function is that used; e.g., in the numerical method of analysis discussed in equations (5.6), (5.7) and (5.8). It is, of course, a straightforward technique.

[39] C. Cox and W. H. Munk [1952, 1953]: "The Measurement of the Roughness of the Sea Surface from Photographs of the Sun's Glitter." Part I, Part II and Part III. AF Technical Reports No. 1, 2, and 3, S.I.O. References 52-61, 53-53 and 54-10.

The method described by Mr. Fuchs for predicting the response of a ship by means of a kernel function is complementary to the method presented in the paper. Application of the method to irregular waves and to confused seas has to be worked out and reduced to simple, straightforward operations which can be applied with confidence. The development of this method, only recently initiated, is promising and should be followed with interest. It appears that in the end it will be necessary to introduce the energy spectrum of the seaway if predictions of motions are to be made.

Dr. Neumann's remarks have served to clarify some important points which received inadequate presentation in the text. In particular, his comments on the meaning of the term "confused" when applied to the seaway are helpful in removing some misunderstanding which may have resulted from use of this word. This term, which is not original with the authors, was used to denote such seas that are formed by the combination of irregular wave trains coming from more than one direction. Dr. Neumann's comments on the present state of oceanographic knowledge of waves reveal to what extent the oceanographer can help the naval architect in pursuing the lines of research suggested in the paper.

Without the interest and initial help of Professor Tukey it is quite possible that the paper would not have been started. He should like the point properly emphasized that it is not a single seaway that represented by equation (1.10) but a whole ensemble of seaways having in common certain statistical properties derivable from an energy spectrum. The point cannot be repeated too often. However, in a sense, by virtue of the ergotic theorem equation (1.10) can be thought of as representing a particular realization of the seaway. The statistical properties of this particular realization can be deduced as a function of time and space.

The clarification of the meaning of the square-root sign over the differential is important and helpful to understand the meaning of the strange expressions used.

It is heartening to learn Professor Tukey's opinion that the study of phases will not increase the complexity of the problem severely. Consequently, it appears that the important problem of slamming, which is essentially a phase-relation problem, may soon yield to the same technique. It was quite surprising to learn that some nonlinear aspects of the problem also can be treated, for nonlinearity is a formidable obstacle to most theoretical developments.

In accordance with Professor Tukey's remarks,

we have deleted all references to Lebesque in the text and in the comments.

Mr. Marks has described briefly the trial work under way at Woods Hole which we hope will serve to verify the statistical approach of the paper. When completed, the work should provide a most valuable evaluation of the present statistical approach. The authors hope that when evaluated their concept of ship responses to confused seas will not be found to be too wanting in realism.

PRESIDENT BLEWETT: We thank Dr. Pierson and Mr. St. Denis for the paper. To me the word "dynamonumental" would probably be the best way to describe this paper. I am sure that it will be used as a reference for years to come.

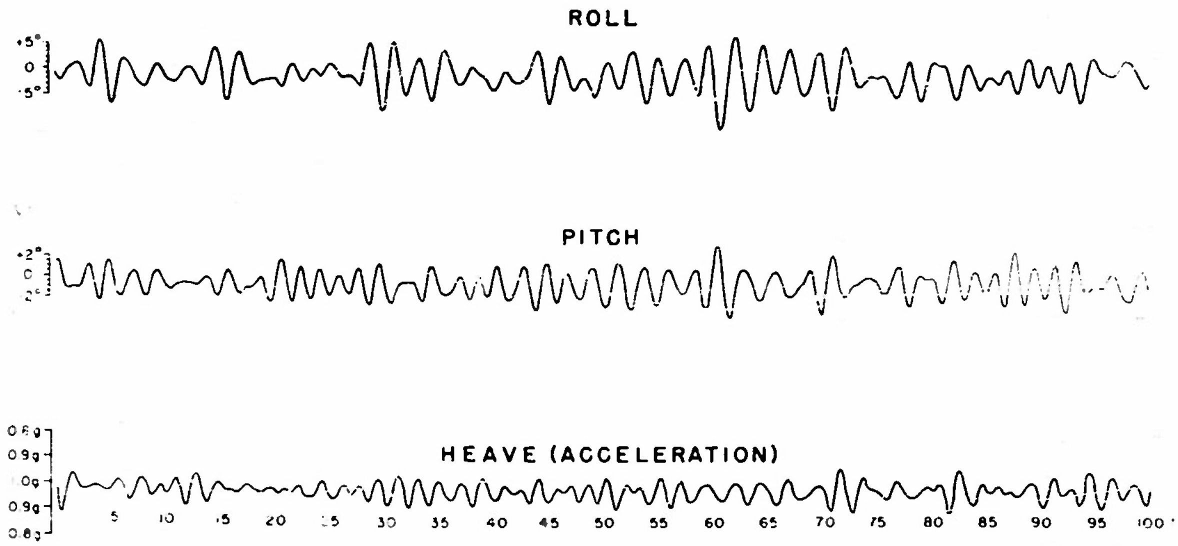

FIG. 2—MOTIONS OF THE LAUNCH RISK IN A HEAD SEA 2-2½ FT HIGH

(To accompany Dr. Wilbur Marks discussion on page 352.)

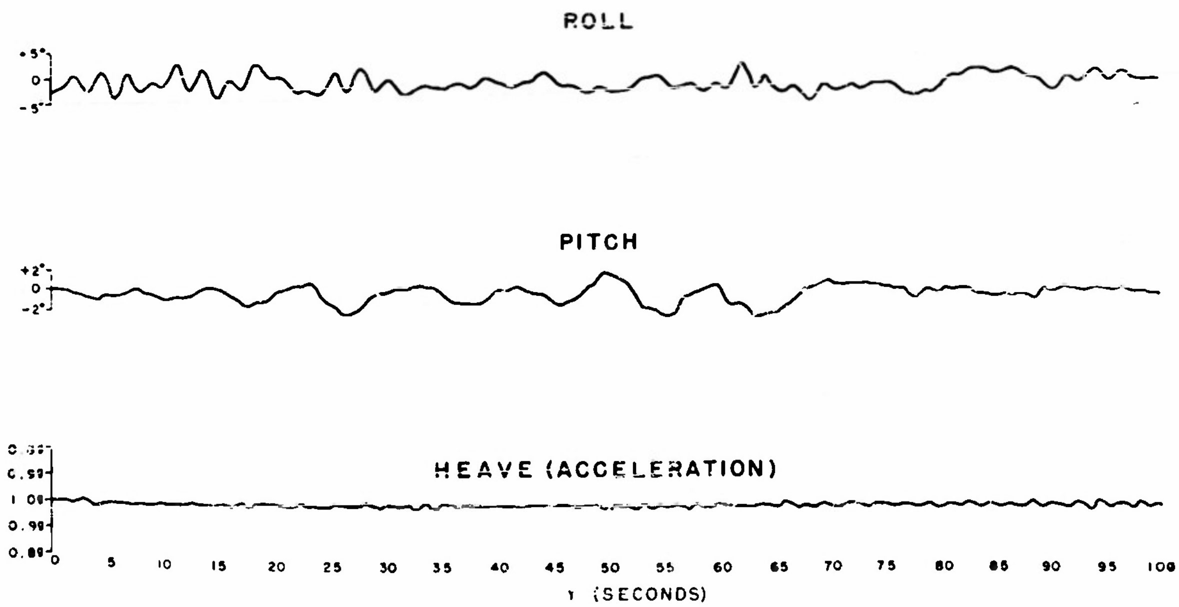

FIG. 3—MOTIONS OF THE LAUNCH RISK IN A FOLLOWING SEA 2-2½ FT HIGH

(To accompany Dr. Wilbur Marks discussion on page 352)